Generalizing the DGLAP Evolution of Fragmentation Functions to the Smallest x Values

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Describe inclusive hadron production, ... $\to h + X$, with a single formalism valid at large *and* small x_p

1. Fixed Order Evolution
2. Double Logarithms
3. Resumming DLs in DGLAP Evolution
4. Connection to the MLLA
5. Fitting to Data
6. Conclusions

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[1] S. Albino, BAK, G. Kramer, W. Ochs, PRL95(2005)232002, PRD73(2006)054020.

1. Fixed Order Approach

- Perturbative description of $e^+e^- \to h + X$

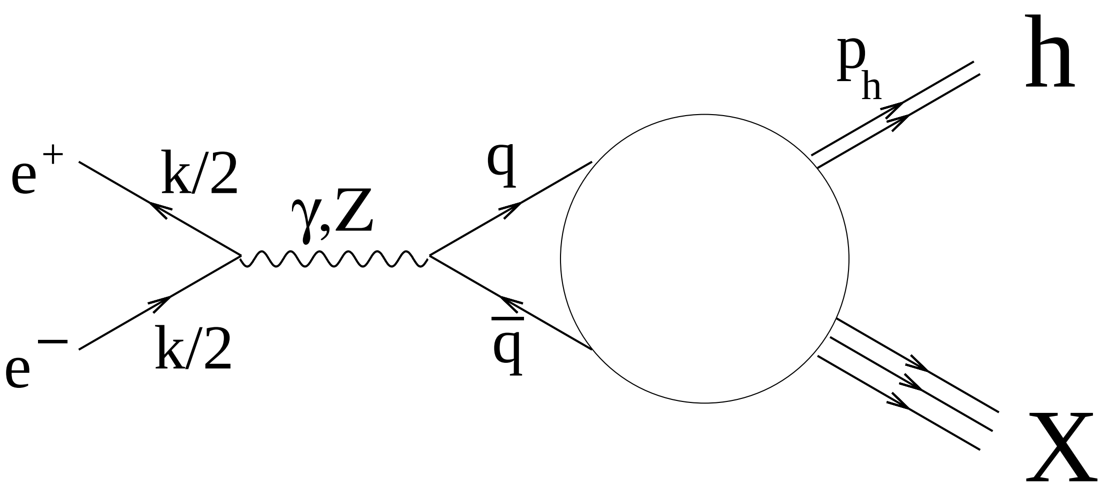

Figure : Diagrammatic representation of $\frac{d\sigma^h}{dx_p}(x_p, s)$

- $x_p = \frac{2p_h}{\sqrt{s}} \quad (0 \leq x_p \leq 1) \qquad \sqrt{s} = \sqrt{k^2}$

- XS to produce h = XS to produce parton i $\times$ Probability that i emits h

or

$$\frac{d\sigma^h}{dx_p}(x_p, s) = \sum_i \int_{x_p}^1 \frac{dy}{y} \frac{d\sigma^i}{d(x_p/y)} \left(\frac{x_p}{y}, s, Q^2\right) D_i^h(y, Q^2)$$

• Total XS independent of *factorization scale* Q

• $D_i^h(y, Q^2)$ are the *fragmentation functions*, probability that i emits h at momentum fraction y

• $\frac{d\sigma^i}{d(x_p/y)}\left(\frac{x_p}{y}, s, Q^2\right)$ perturbatively calculable as series in $a_s = \frac{\alpha_s}{2\pi}$.
Keep $Q = O(\sqrt{s})$ to make $a_s^n \ln^m \frac{Q^2}{s}$ terms small

• $\therefore$ need Q dependence of $D_i^h(y, Q^2)$

• Evolution of FFs in factorization scale Q described by DGLAP equation

$$\frac{d}{d\ln Q^2} D(x, Q^2) = \int_x^1 \frac{dy}{y} P(y, a_s(Q^2)) D\left(\frac{x}{y}, Q^2\right)$$

$(D = (D_g, D_d, D_{\bar{d}}, D_u, ...))$

• P as series in a_s is good approximation at large x.

• "large x": Data can be well described down to $x = 0.1$.
E.g. analyses of KKP
(Kniehl, Kramer, Pötter - Nucl. Phys. B582 (2000) 514)
or AKK
(Albino, Kniehl, Kramer - Nucl. Phys. B725 (2005) 181; B734 (2006) 50),

• FFs parameterized as

$$D_i = N x^{\alpha} (1 - x)^{\beta}$$

and N, α, β for each FF
and also $\alpha_s(M_Z)$ are fitted to e^+e^- inclusive data
for light charged hadron ($\pi^{\pm}$, $K^{\pm}$, $p/\bar{p}$) production
from ALEPH, DELPHI, OPAL, SLD and TPC.

• Gives competitive $\alpha_s(M_Z)$

• FFs well constrained
and lead to good description of hadron production data
from pp and $p\bar{p}$ collisions (e.g. PHENIX, UA1, STAR)
provided p_T is not too low.

2. Double Logarithms

• Procedure of FF determination
has some similarities with global fits of PDFs

• However, PDFs determined
down to much smaller x (10^{-3} or 10^{-4})
since singularities for $x \to 0$ in spacelike case,

$$P_{qq}, P_{qg} \quad \to \quad 0, \quad \alpha_s^2 \frac{1}{x}$$

$$P_{gq}, P_{gg} \quad \to \quad \alpha_s \frac{1}{x}, \quad \alpha_s^2 \frac{1}{x}$$

weaker than those in timelike splitting functions

$$P_{qq}, P_{qg} \quad \to \quad 0, \quad \alpha_s^2 \frac{1}{x}$$

$$P_{gq}, P_{gg} \quad \to \quad \alpha_s \frac{1}{x}, \quad \alpha_s^2 \frac{\ln^2 x}{x}$$

• Want to describe small (< 0.1) x data at the same time.

So,

• Splitting function $P(x, a_s)$ at LO, $a_s P^{(0)}(x)$,
contains $x \to 0$ divergence a_s/x,
a *double logarithm* (DL)

• At higher orders in the splitting function,
DLs are of the form
$(1/x)(a_s \ln x)^2 (a_s \ln^2 x)^r$, $r = -1, ..., \infty$

• So DGLAP equation
in fixed order approach (series in a_s)
is bad approximation for $\ln(1/x) = O(a_s^{-1/2})$

• At small x, evolution better described by
Double Logarithmic Approximation (DLA),
which accounts for all DLs in the evolution

• DLA equation accounts for all DLs,
which arise from tree level gluon emission off quark/antiquark line

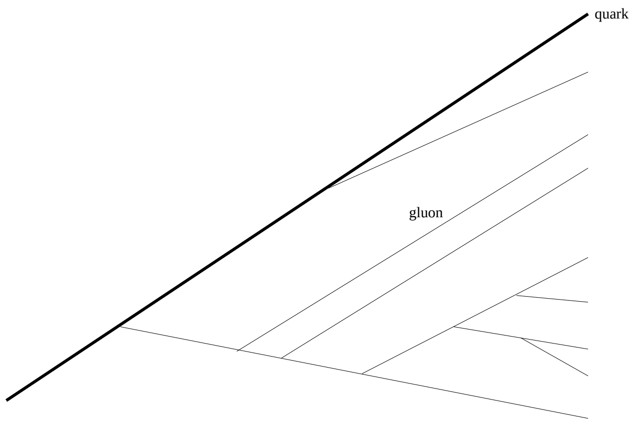

Figure : Diagram containing DLs

• $(i-1)$th gluon emits ith gluon, probability $\propto$

$$a_s \frac{d\theta_i}{\theta_i} \frac{dE_i}{E_i} \propto a_s \ln^2$$

• To pick out largest logarithms,
strongly order energies and emission angles of gluons,
$(E,\theta)_i \ll (E,\theta)_{i-1}$

3. Resumming DLs in DGLAP Evolution

- DLA equation

$$\frac{d}{d\ln Q^2} D(x,Q^2) = \int_x^1 \frac{dy}{y} \frac{2C_A}{y} A a_s(y^2Q^2) D\left(\frac{x}{y}, y^2Q^2\right)$$

where

$$A = \begin{pmatrix} 0 & \frac{2C_F}{C_A} \\ 0 & 1 \end{pmatrix} \tag{1}$$

for $D = (D_\Sigma, D_g)$ ($D_\Sigma = \frac{1}{n_f}\sum_{q=1}^{n_f}(D_q + D_{\bar{q}})$),
while $A = 0$ for valence and non-singlet quark FFs

- Use DLA equation to obtain complete DL contribution to splitting function, P^{DL}

- Then in DGLAP equation (at LO), replace

$$a_s P^{(0)}(x) \longrightarrow a_s \overline{P}^{(0)}(x) + P^{\mathrm{DL}}(x, a_s),$$

where $a_s\overline{P}^{(0)}(x)$ is $a_sP^{(0)}(x)$
after DL (a_s/x) has been subtracted
to prevent double counting

• To obtain $P^{\rm DL}$ from DLA equation, work in Mellin space,

$$f(\omega) = \int_0^1 dx x^\omega f(x)$$

• DLs in Mellin space are of the form $(a_s/\omega)(a_s/\omega^2)^{r+1}$, i.e. singularities as $\omega \to 0$

• DLA equation in Mellin space reads

$$\left(\omega + 2\frac{d}{d\ln Q^2}\right)\frac{d}{d\ln Q^2} D(\omega, Q^2) = 2C_A a_s(Q^2) A D(\omega, Q^2)$$

• Insert DGLAP equation in Mellin space (accurate at small ω)

$$\frac{d}{d\ln Q^2} D(\omega, Q^2) = P^{\rm DL}(\omega, a_s(Q^2)) D(\omega, Q^2),$$

then DLA equation without higher order terms reads

$$2(P^{\rm DL})^2 + \omega P^{\rm DL} - 2C_A a_s A = 0$$

- Choose solution

$$P^{\mathrm{DL}}(\omega, a_s) = \frac{A}{4}\left(-\omega + \sqrt{\omega^2 + 16C_A a_s}\right),$$

since its expansion in a_s yields

$$a_s P^{\mathrm{DL}(0)}(\omega, a_s) = \begin{pmatrix} 0 & a_s\frac{4C_F}{\omega} \\ 0 & a_s\frac{2C_A}{\omega} \end{pmatrix}, \qquad (2)$$

which agrees with LO (and NLO) DL in literature

- This result contains all DLs in splitting function

- This result in x space reads

$$P^{\mathrm{DL}}(x, a_s) = \frac{A\sqrt{C_A a_s}}{x \ln\frac{1}{x}} J_1\left(4\sqrt{C_A a_s}\ln\frac{1}{x}\right)$$

- As $x \to 0$, $P^{\mathrm{DL}}(x, a_s) \to \frac{1}{x\ln^{\frac{3}{2}}\frac{1}{x}}$

$\longrightarrow$ Less than LO singularity $\frac{\ln^2\frac{1}{x}}{x}$

• To summarize, evolve FFs via

$$\frac{d}{d\ln Q^2} D(x, Q^2) = \int_x^1 \frac{dy}{y} P(y, a_s(Q^2)) D\left(\frac{x}{y}, Q^2\right)$$

but for P use

$$a_s P^{(0)}(x) \longrightarrow a_s \overline{P}^{(0)}(x) + P^{\rm DL}(x, a_s)$$

where $P^{\rm DL}$ contains complete DL contribution,
$a_s \overline{P}^{(0)}(x)$ is $a_s P^{(0)}(x)$
after DLs (of form a_s/x at LO) are subtracted

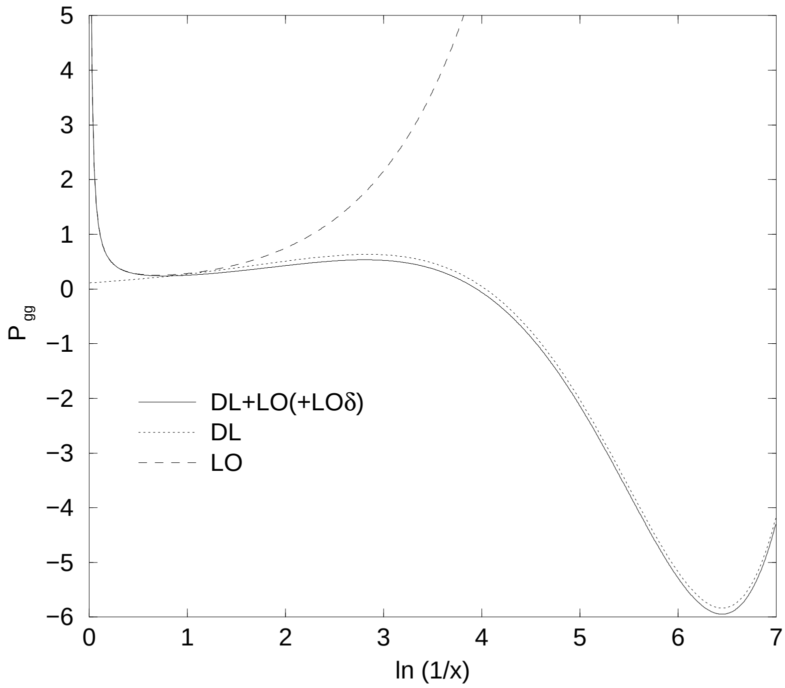

Figure : (i) $P_{gg}(x, a_s)$ calculated in the DL+LO(+LOδ) scheme, (ii) $P_{gg}(x, a_s)$ calculated to $O(a_s)$ in the FO approach (labelled "LO"), and (iii) $P_{gg}^{\rm DL}(x, a_s)$ (labelled "DL"). $a_s = 0.118/(2\pi)$.

• Note: DLs are the most singular *soft gluon logarithms*. The next SGL, the *single logarithm*, is present at LO. However, this SL is of $O(a_s\omega^0)$
$\longrightarrow$ does not need to be resummed.

• At NLO, a number of SGLs occur (DL, SL, ...) which must be resummed (since they are singular).

• Our method can be applied at higher orders
by subtracting all SGLs,
then adding in the complete DL contribution,
neglecting SLs and smaller SGLs,
since one assumes their resummed contributions are small relative to the DL one.

• Generally,

$$P(x,a_s)=\sum_{n=1}^{\infty} a_s^n P^{\mathrm{FO}(n-1)}(\omega)+\sum_{m=1}^{\infty}\left(\frac{a_s}{\omega}\right)^m g_m\left(\frac{a_s}{\omega^2}\right) \tag{3}$$

• $P^{\mathrm{FO}(n-1)}(\omega)$ finite for $\omega\to 0$

• $m=1$ term contains all DLs

• $m=2$ term contains all SLs

4. Connection to the MLLA

• This approach is equivalent to adding in remaining fixed order contribution to DLA equation,

$$\frac{d}{d\ln Q^2}D(\omega,Q^2) = \left(\omega + 2\frac{d}{d\ln Q^2}\right)^{-1} 2C_A a_s(Q^2) A D(\omega,Q^2)$$
$$+ a_s(Q^2)\overline{P}^{(0)}(\omega) D(\omega,Q^2)$$

• This DLA+FO equation reproduces the Modified Leading Logarithmic Approximation (MLLA) by making two approximations:

1. Approximate $a_s\overline{P}^{(0)}(\omega)$ by $a_s\overline{P}^{(0)}(\omega=0)$, which equals the LO *single logarithm* (SL),

$$P^{\mathrm{SL}(0)}(\omega) = \begin{pmatrix} 0 & -3C_F \\ \frac{2}{3}T_R n_f & -\frac{11}{6}C_A - \frac{2}{3}T_R n_f \end{pmatrix} \tag{4}$$

2. Use small ω DLA result

$$D_{q,\overline{q}} = \frac{C_F}{C_A} D_g$$

Then gluon component of DLA+FO equation is the MLLA equation

- MLLA equation:

$$\left(\omega + 2\frac{d}{d\ln Q^2}\right)\frac{d}{d\ln Q^2}D_g(\omega, Q^2) = 2C_A a_s(Q^2) D_g(\omega, Q^2)$$
$$-\left(\omega + 2\frac{d}{d\ln Q^2}\right) a_s(Q^2)\frac{a}{2}D_g(\omega, Q^2)$$

$(a = \frac{11}{3}C_A + \frac{4}{3}T_R n_f \left(1 - \frac{2C_F}{C_A}\right))$

- But we want to keep quark and gluon freedom
- and large x (ω) behaviour of P

5. Fitting to Data

- Perform fits using usual LO DGLAP evoluition

$$P \approx a_s P^{(0)}$$

and then again with DLs resummed

$$P \approx a_s \overline{P}^{(0)} + P^{\mathrm{DL}}$$

- Fit to charged hadron data
(constraining quark FFs)

- At LO,

$$\frac{1}{\sigma(s)} \frac{d\sigma}{dx_p}(x_p, s) = \frac{1}{n_f \langle Q(s) \rangle} \sum_q Q_q(s) D_q^+(x_p, s)$$

- Note we have chosen $Q = \sqrt{s}$ (usual in DGLAP fits)

- Fit FFs $D(x, Q_0^2)$ at some initial scale Q_0

- FFs to be fitted are

$$
\begin{aligned}
&1. D_g(x, Q_0^2), \\
&2. D_{uc}(x, Q_0^2) = \frac{1}{2}\left(D_u(x, Q_0^2) + D_c(x, Q_0^2)\right), \\
&3. D_{dsb}(x, Q_0^2) = \frac{1}{3}\left(D_d(x, Q_0^2) + D_s(x, Q_0^2) + D_b(x, Q_0^2)\right)
\end{aligned}
$$

(quarks of identical charge
cannot be distinguished by the data)

- Note from charge conjugation symmetry
(since hadron charges are not measured),

$$D_{\bar{q}} = D_q$$

- Choose $Q_0 = 14$ GeV,
as $\sqrt{s} = 14$ GeV is the lowest $\sqrt{s}$ data

- Set $n_f = 5$

- Only fit to data for which $\xi < \ln \sqrt{s}$

- For each FF, choose parameterization

$$D_i(x, Q_0^2) = N \exp[-c \ln^2 x] x^\alpha (1-x)^\beta$$

- At large and intermediate x,

$$D_i(x, Q_0^2) \approx N x^\alpha (1-x)^\beta$$

- At small x

$$\lim_{x \to 0} D_i(x, Q_0^2) = N \exp[-c \ln^2 \frac{1}{x} - \alpha \ln \frac{1}{x}];$$

$c > 0$: Gaussian in $\ln(1/x)$ of width $1/\sqrt{2c}$,
centre at $-\alpha/(2c)$ (therefore must have $\alpha < 0$)
Normalization $N\sqrt{\pi/c}\exp\left[\alpha^2/(4c)\right]$

For large Q_0, small x, this is the behaviour we expect from DLA

- Recall at small x that

$$D_{q,\bar{q}} = \frac{C_F}{C_A} D_g$$

So we expect it is a good approximation to set

$$c_{uc} = c_{dsb} = c_g,$$
$$\alpha_{uc} = \alpha_{dsb} = \alpha_g$$

- If we just fitted to small x data, we could also choose

$$N_{uc} = N_{dsb} = \frac{C_F}{C_A} N_g,$$

but we want to describe large x data too

• First do standard LO DGLAP fit, i.e. with

$$P \approx a_s P^{(0)}$$

Table : Parameter values for the FFs at $Q_0 = 14$ GeV from a fit to all data using DGLAP evolution in the FO approach to LO. $\Lambda_{\rm QCD} = 388$ MeV. $\chi^2_{\rm DF} = 3.0$

FF \ Parameter	N	β	α	c
g	0.22	−0.43	−2.38	0.25
u+c	0.49	2.30	[−2.38]	[0.25]
d+s+b	0.37	1.49	[−2.38]	[0.25]

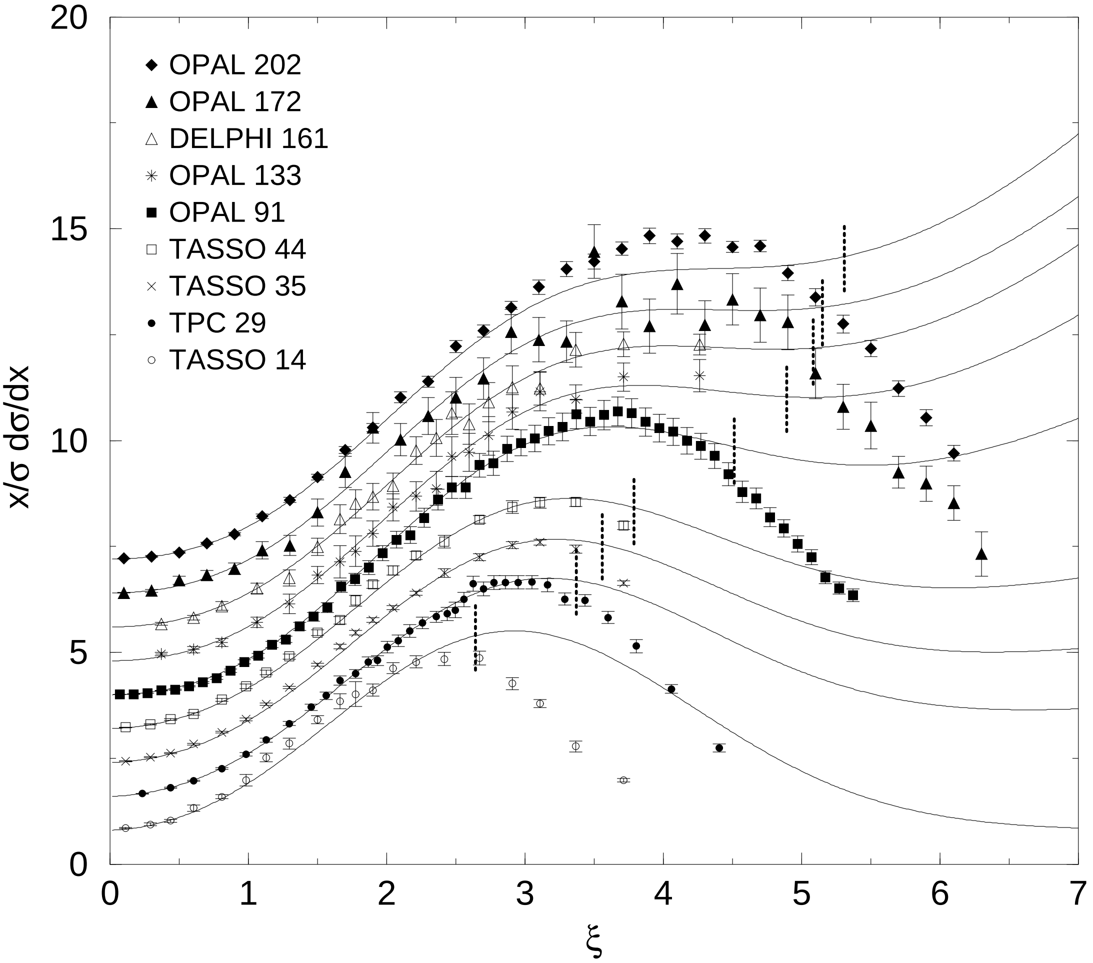

Figure : Fit to data with usual LO DGLAP evolution. Each curve is shifted up by 0.8 for clarity. $\xi = \ln \frac{1}{x}$

• Now fit with DLs resummed, i.e. with

$$P \approx a_s \overline{P}^{(0)} + P^{\mathrm{DL}}$$

Table : Parameter values for the FFs at $Q_0 = 14$ GeV from a fit to all data using DGLAP evolution in the DL+LO+LOδ scheme. $\Lambda_{\mathrm{QCD}} = 801$ MeV. $\chi^2_{\mathrm{DF}} = 2.1$

FF \ Parameter	N	β	α	c
g	1.60	5.01	-2.63	0.35
u+c	0.39	1.46	[-2.63]	[0.35]
d+s+b	0.34	1.49	[-2.63]	[0.35]

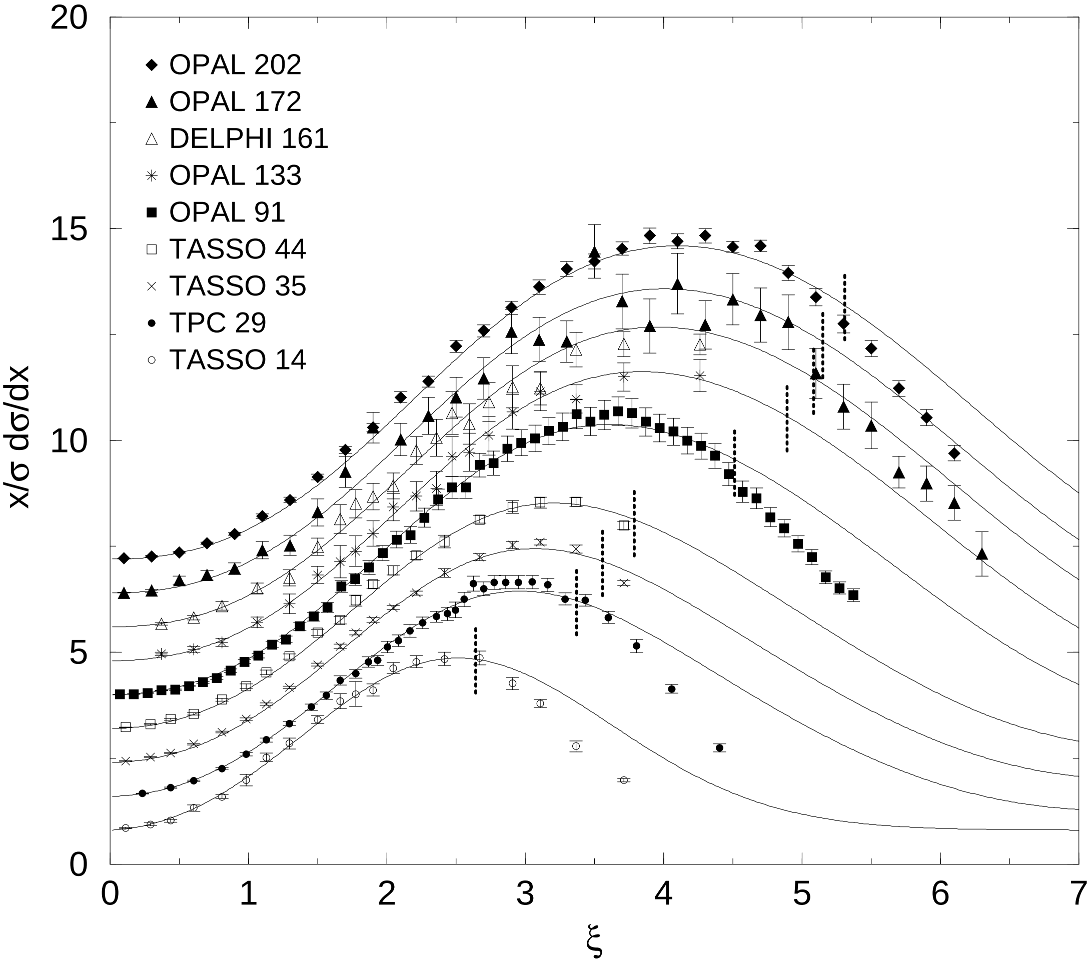

Figure : As before, but now DLs in the evolution are resummed. $\xi = \ln \frac{1}{x}$

• Resummation: significant decrease in χ^2_{DF}, 3.0 $\longrightarrow$ 2.1.
Much better description of data around the peak,
but description beyond the peak not sufficient

• Big increase in Λ_{QCD}, 400 MeV $\longrightarrow$ 800 MeV
We chose $Q=\sqrt{s}$.
Had we made MLLA choice of $Q=\sqrt{s}/2$,
the only difference in our fit would be to halve our Λ_{QCD},
800 MeV $\longrightarrow$ 800/2 MeV
(Q always appears in the ratio Q/Λ_{QCD} at LO)

• Recall at small x

$$N_{uc}=N_{dsb}=\frac{C_F}{C_A}N_g,$$

N_g should only be about double N_{uc} or N_{dsb},
but it is two times too big

• Hadron mass effects important at small x

• Work with light cone coordinates, $V = (V^+, V^-, \mathbf{V}_T)$, where $V^\pm = \frac{1}{\sqrt{2}}(V^0 \pm V^3)$ and $\mathbf{V}_T = (V^1, V^2)$

• Momentum of the electroweak boson in CM frame:

$$q = \left(\frac{\sqrt{s}}{\sqrt{2}}, \frac{\sqrt{s}}{\sqrt{2}}, \mathbf{0} \right)$$

Momentum of hadron, mass m_h, moving along 3-axis:

$$p_h = \left(\frac{\eta\sqrt{s}}{\sqrt{2}}, \frac{m_h^2}{\sqrt{2}\eta\sqrt{s}}, \mathbf{0} \right)$$

• η is like *Nachtmann scaling variable* of DIS, $\eta \to 1/\eta$, Lorentz invariant w.r.t. boost along 3-axis

• η is true scaling variable

$$x_p = \eta \left(1 - \frac{m_h^2}{s\eta^2} \right)$$

so mass effects only important at small x_p (η)

• Neglect hadron mass, ($m_h \ll x_p\sqrt{s}$), then $x_p \approx \eta$

• At leading twist, after factorization,
parton of momentum k fragments into hadron:

$$k = \left(\frac{p_h^+}{y}, 0, \mathbf{0} \right)$$

• Kinematics: $\eta < y < 1$

$$\frac{d\sigma}{d\eta}(\eta, s) = \int_\eta^1 \frac{dy}{y} \frac{d\sigma}{dy}(y, s, Q^2) D\left(\frac{\eta}{y}, Q^2 \right)$$

• Experimentalists measure

$$\frac{d\sigma}{dx_p}(x_p, s) = \frac{1}{1 + \frac{m_h^2}{s\eta^2(x_p)}} \frac{d\sigma}{d\eta}(\eta(x_p), s)$$

• Redo fits, fit m_h

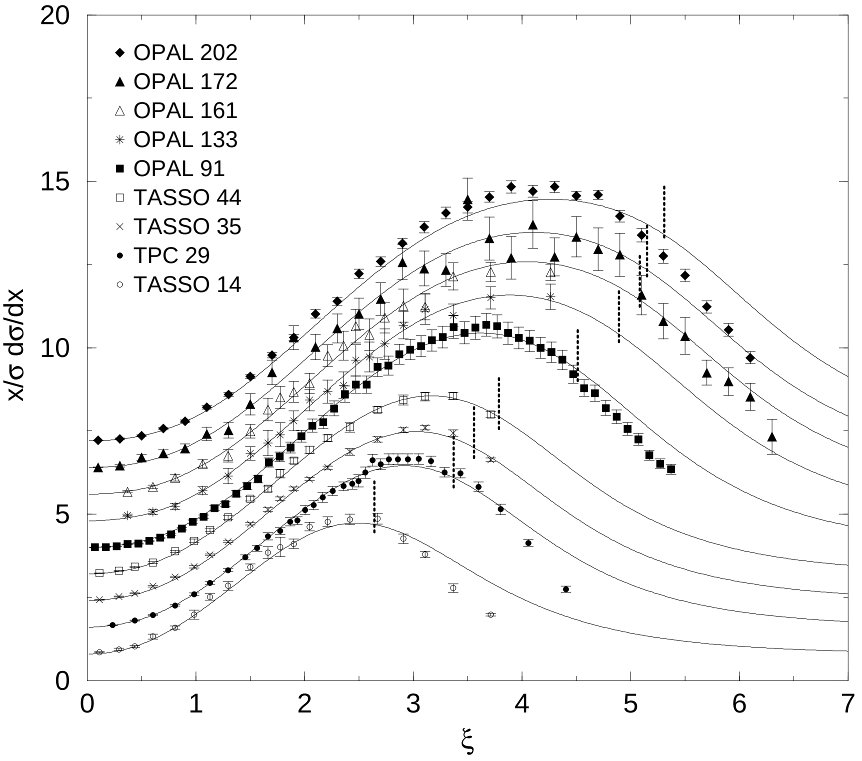

Figure : Unresummed fit with hadron mass effects

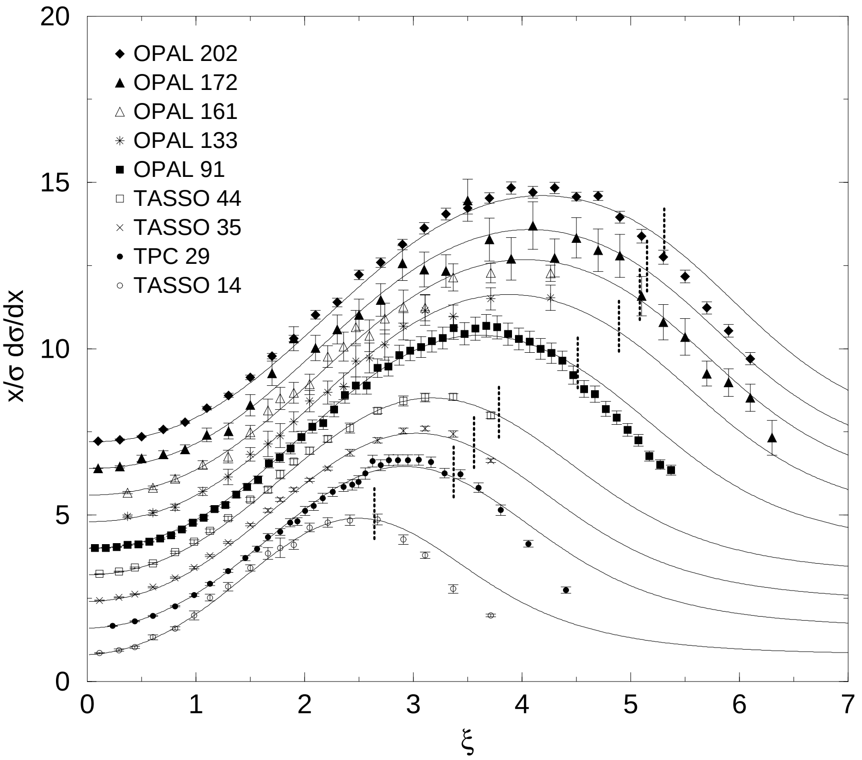

Figure : Resummed fit with hadron mass effects

- Now *both* fits give $\chi^2_{\rm DF} \approx 2$

- In both cases, $m_h \approx 300$ MeV,
which is reasonable for sum over $\pi^{\pm}$, $K^{\pm}$, $p/\overline{p}$

- However, unresummed case: $\Lambda_{\rm QCD} = 1300$ MeV

- Resummed case: $\Lambda_{\rm QCD} = 400$ MeV

- No significant improvement to description of gluon data

• Now fit with DLs resummed, m_h finite, and OPAL gluon data included. $m_h = 302$ GeV, $\chi^2_{\rm DF} = 2.1$.

Table : Parameter values for the FFs at $Q_0 = 14$ GeV from a fit to all data, including the OPAL gluon jet data, using DGLAP evolution in the DL+LO+LOδ scheme and with mass effects incorporated. $\Lambda_{\rm QCD} = 490$ MeV.

FF \ Parameter	N	β	α	c
g	1.30	5.09	-2.30	0.24
$u+c$	0.46	1.70	$[-2.30]$	$[0.24]$
$d+s+b$	0.53	1.75	$[-2.30]$	$[0.24]$

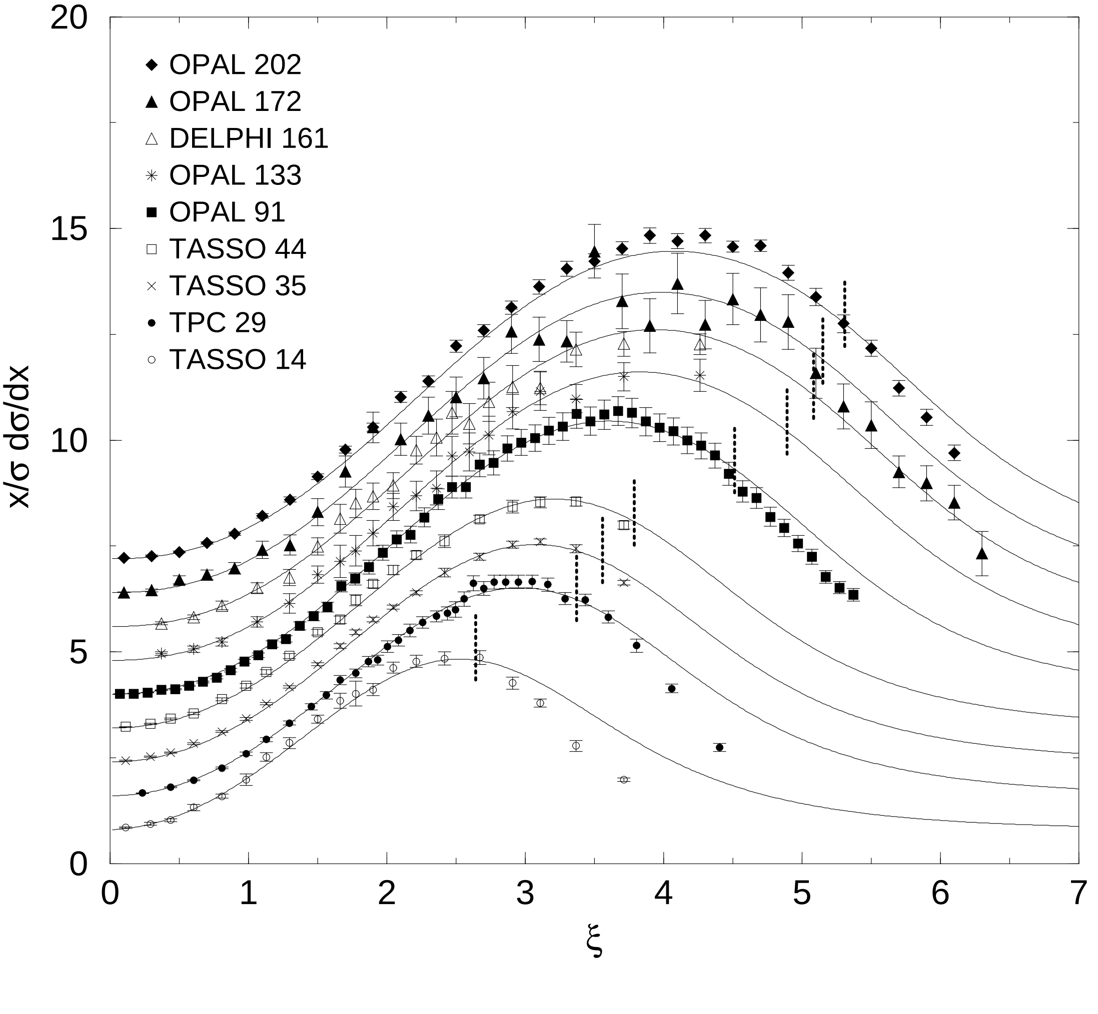

Figure : Resummed fit with hadron mass effects and OPAL gluon data. $\xi = \ln \frac{1}{x}$

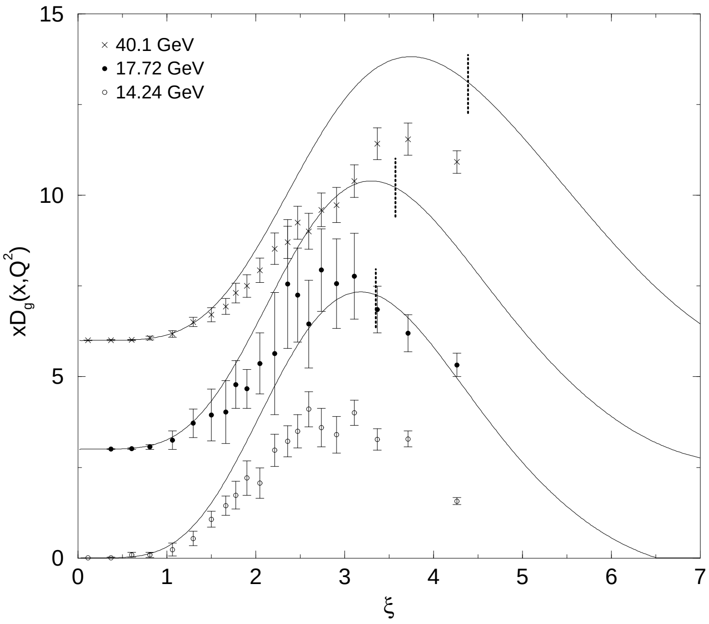

Figure : Comparison of OPAL gluon data with gluon FF from fit where these data are not included and hadron mass effects are neglected.

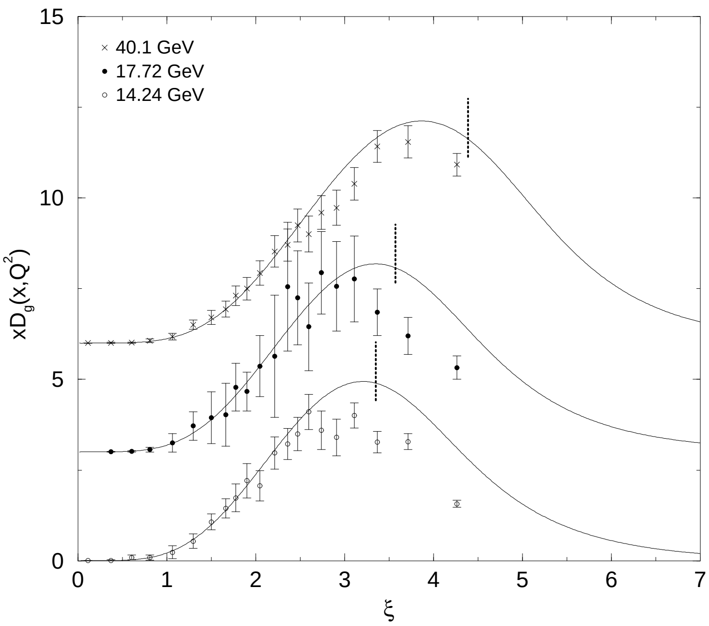

Figure : Now OPAL gluon data and hadron mass effects included in the fit.

6. Conclusions

1. Extended DGLAP evolution to small x_p via DLA equation

2. Performed fits to data and found much improvement in the peak region compared to FO DGLAP

3. Our scheme reproduces the MLLA by making certain approximations, but it is more general

4. Treatment of hadron mass effects lead to more sensible value of $\Lambda_{\rm QCD}$

Further work:

- Apply to NLO global fits to extend range to smaller x_p (currently $0.1 < x_p < 1$)

- Study other small x effects (SLs?) to improve large ξ region yet further