T-odd functions in relativistic eikonal models

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Generalizations of collinear Parton Distributions

GPDs (off-diagonal):
measurable in exclusive processes
provides 3-dim. spatial picture

TMDs (small transv. deviations):
measurable in semi-inclusive processes
provides 3-dim. momentum picture

$$\Phi_{ij}(x) = \int_{-\infty}^{\infty} \frac{dz^-}{2\pi} \mathrm{e}^{ixP^+z^-} \langle P| \bar{q}_j(-\tfrac{z^-}{2}) \, [-\tfrac{z^-}{2}; \tfrac{z^-}{2}] \, q_i(\tfrac{z^-}{2}) \, |P\rangle$$

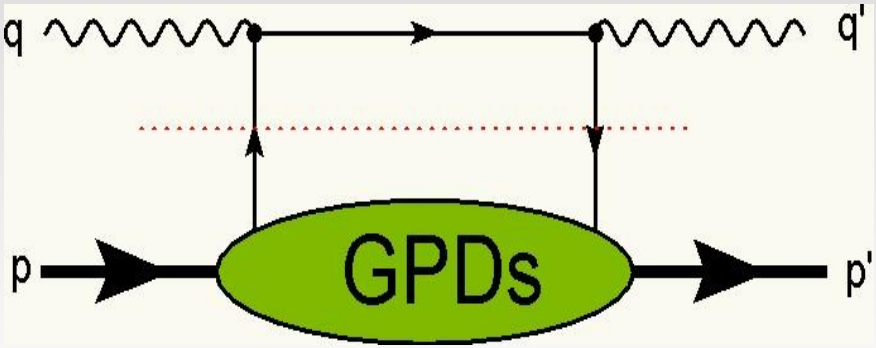
Collinear PDFs $f_1(x,\mu), g_1(x,\mu), h_1(x,\mu)$

Theory well-understood, delivers a one-dimensional picture of nucleon structure

Measurable in DIS (but not sensitive to transversity...)

Generalized Parton Distributions

- Exclusive processes (DVCS, meson production, ...):



$$P = \tfrac{1}{2}(p + p') \qquad \Delta = p' - p$$

$$\Delta^+ = -2\xi P^+ \qquad t = -\frac{4\xi^2 M^2 + \vec{\Delta}_T^2}{1-\xi^2}$$

$$F_{ij}(x,\xi,\vec{\Delta}_T) = \int \frac{dz^-}{2\pi}\, e^{ixP^+z^-} \langle p' | \bar{q}_j(-\tfrac{z^-}{2}) \left[-\tfrac{z^-}{2}; \tfrac{z^-}{2}\right] q_i(\tfrac{z^-}{2}) | p \rangle$$

Eight GPDs → most prominent H and E

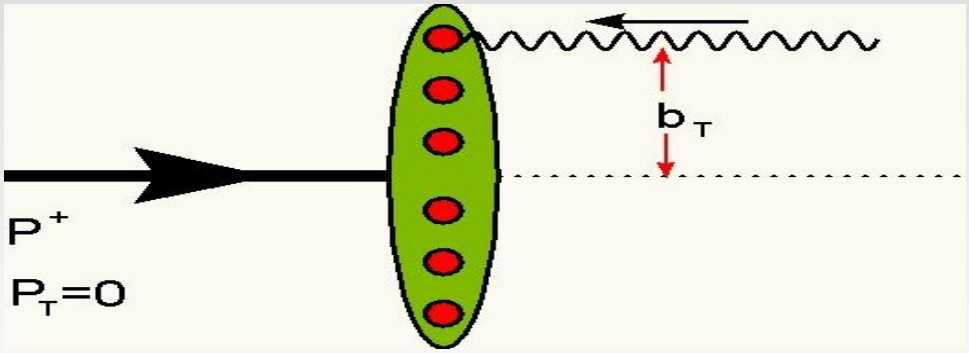
$$P^+ \mathrm{Tr}[F\,\gamma^+] = \bar{u}(p')\left[\gamma^+ H(x,\xi,t) + \frac{i\sigma^{+\mu}\Delta_\mu}{2M} E(x,\xi,t)\right] u(p)$$

E → no correspondence in the collinear limit. Ji-Angular momentum sum rule

$$J^q = \tfrac{1}{2}\int_{-1}^{1} dx\, x[H^q(x,0,0) + E^q(x,0,0)]$$

Impact Parameter Space

- **Impact Parameter Space:** (ξ=0) [M. Burkardt, PRD62, 071503]



- **Impact parameter b_T and transv. momentum transfer Δ_T → FT**

$$\mathcal{F}_{ij}(x,\vec{b}_T) = \int \frac{d^2\Delta_T}{(2\pi)^2}\, \mathrm{e}^{-i\vec{\Delta}_T\cdot\vec{b}_T}\, F_{ij}(x,0,\vec{\Delta}_T)$$

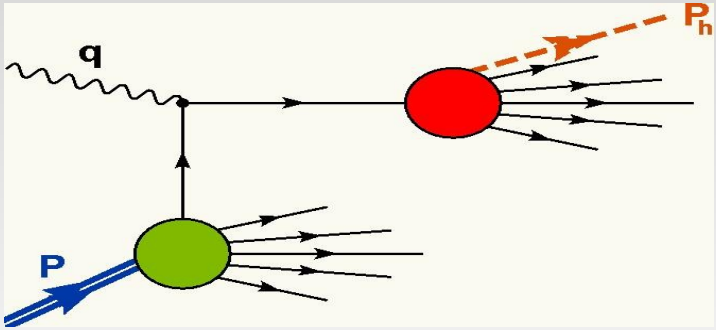
- **Impact parameter space → "diagonal" matrix element** $z_{1/2} = \mp\frac{z^-}{2}n_- + b_T$

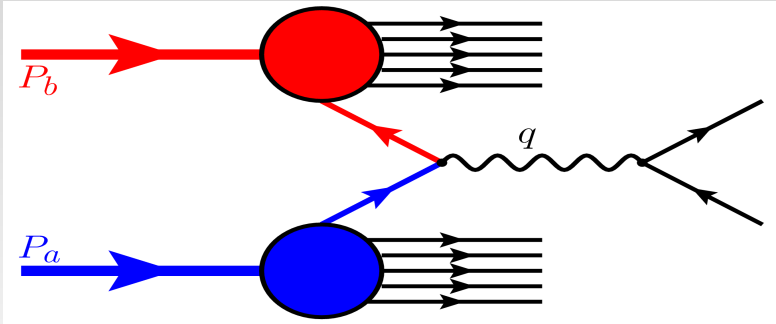
$$\mathcal{F}_{ij}(x,\vec{b}_T) = \int \frac{dz^-}{2(2\pi)}\, \mathrm{e}^{ixP^+z^-}\, \langle P^+;\vec{0}_T|\bar{\psi}_j(z_1)\,[z_1;z_2]\,\psi_i(z_2)|P^+;\vec{0}_T\rangle$$

- **(ξ=0): probability density of partons in transverse plane.**

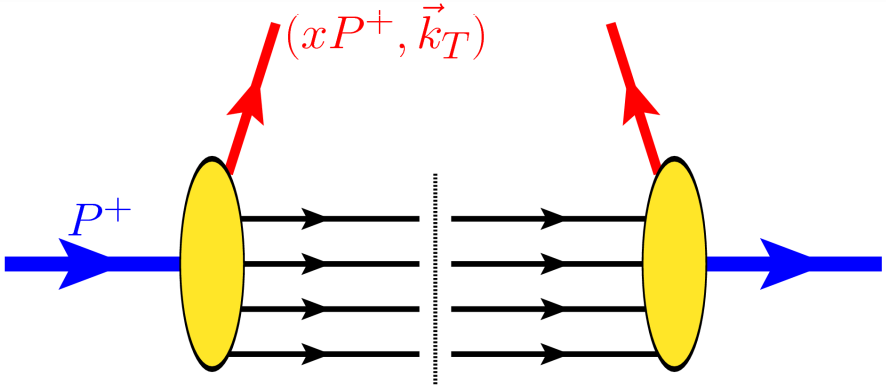
k_T-dependent Parton Distributions (TMDs)

Semi-inclusive processes at small transverse final state momenta:





$$\Phi_{ij}(x,\vec{k}_T;S) = \int \frac{dz^- d^2 z_T}{2(2\pi)^3}\, e^{ik\cdot z} \langle P,S|\,\bar{\psi}_j(0)\mathcal{W}_{\mathrm{SIDIS/DY}}[0,z]\,\psi_i(z)\,|P,S\rangle\Big|_{z^+=0}$$



- **Eight TMDs:**

quark pol.

nucleon pol.	U	L	T
U	f_1		$h_1^\perp$
L		g_1	$h_{1L}^\perp$
T	$f_{1T}^\perp$	g_{1T}	h_1, $h_{1T}^\perp$

Twist-2 TMDs

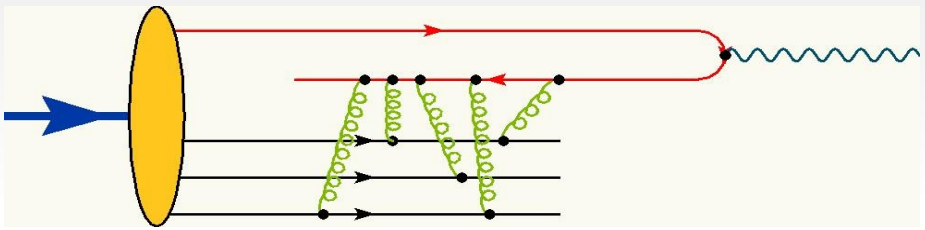
$$\frac{1}{2}\mathrm{Tr}[\Phi\,\gamma^+] = f_1(x,\vec{k}_T^2) - \frac{\varepsilon_T^{ij} k_T^i S_T^j}{M} f_{1T}^\perp(x,\vec{k}_T^2)$$

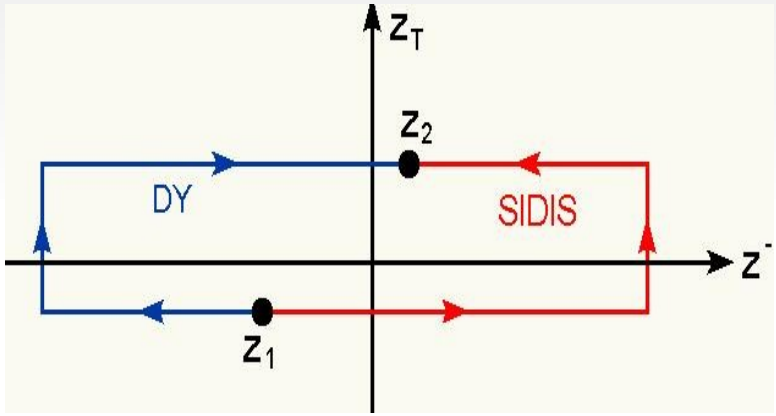
T-odd TMDs

- ***Neglect*** **gauge link operator:**

Time-reversal forbids → **Sivers function**, **Boer-Mulders function**

- **If T-odd TMDs ≠ 0: Gauge Link not neglegible, physical effect:**





$$\mathcal{W}[z_1; z_2] = \mathcal{P}\mathrm{e}^{-ig\int_{z_1}^{z_2} ds\cdot A(s)}$$

Initial / Final state interactions

Time reversal → **switches sign:**

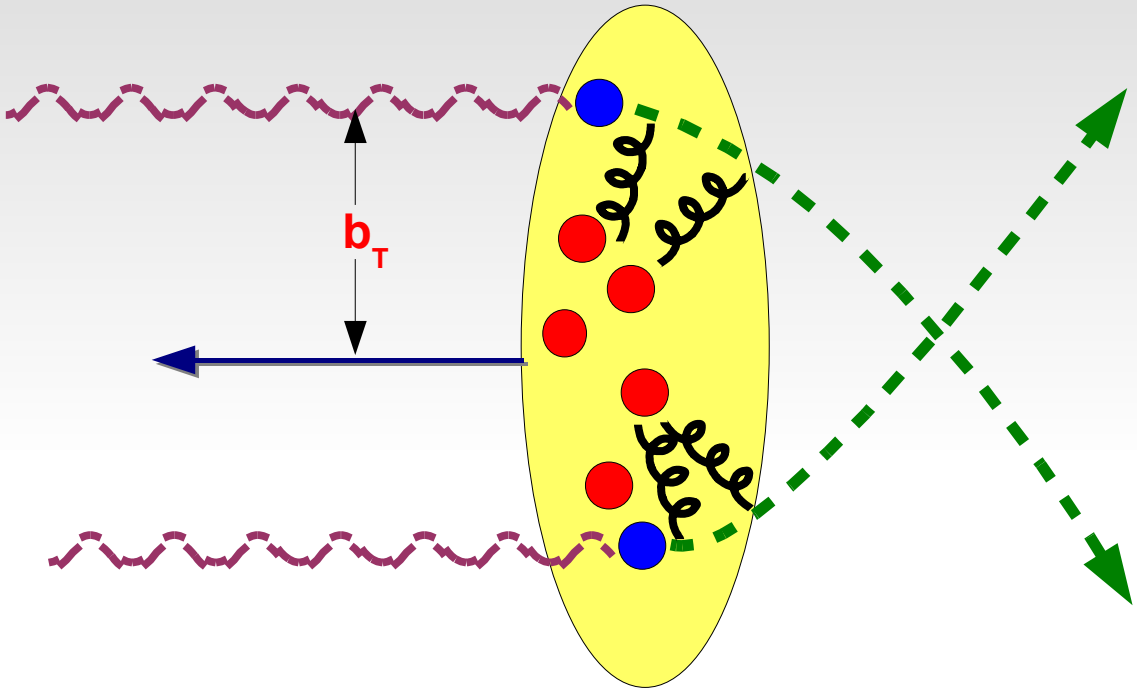
$$f_{1T}^{\perp}\Big|_{DIS} = -f_{1T}^{\perp}\Big|_{DY}$$

$$h_1^{\perp}\Big|_{DIS} = -h_1^{\perp}\Big|_{DY}$$

Illustrative picture of the Final State Interactions

Situation in SIDIS:

Unpolarized nucleon **→ quarks are equally distributed in transverse plane**

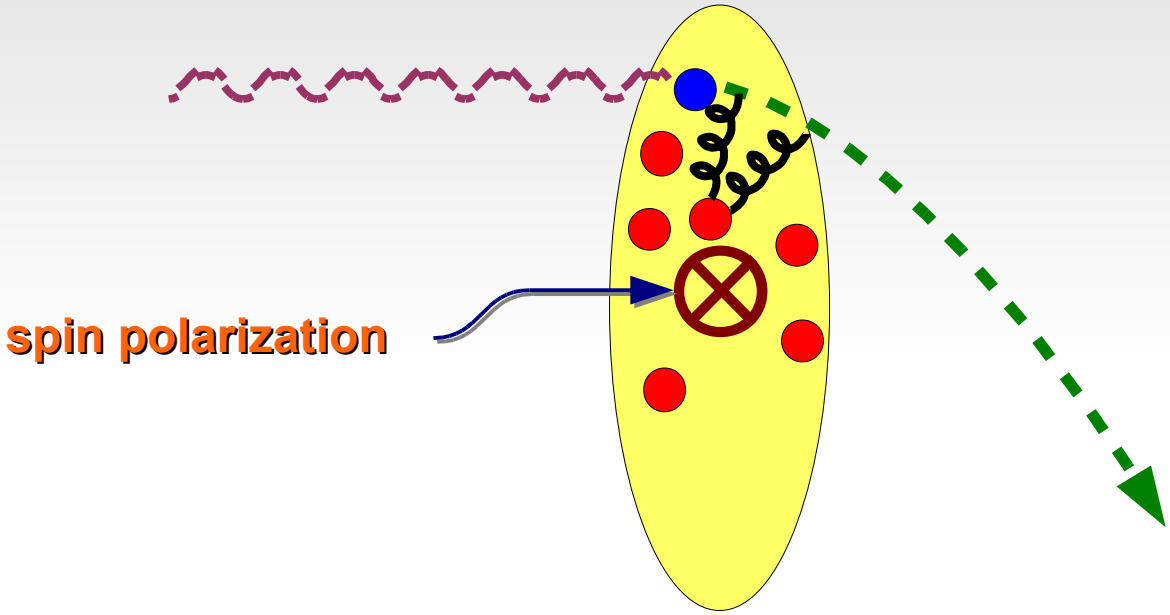


Final State interactions are assumed to be attractive here

Lensing effect, but no observable net effect!

Sivers asymmetry

Transversly polarized nucleon →
spatial distortion of the quark distribution in the transverse plane →
Impact parameter GPD E



Spatial distortion + FSI lead to observable net effect
→ non-zero Left-Right (Sivers) asymmetry
→ attractive FSI: correct prediction of the sign of SSA

Relation Sivers – E

Non-trivial relations for "T-odd" parton distributions:

M. Burkardt [Nucl.Phys. A735, 185], [PRD66, 114005]

Average transverse momentum of unpolarized partons in a transversely polarized nucleon:

$$\langle k_T^i \rangle_T(x) = \int d^2k_T\, k_T^i\, \frac{1}{2}\Big[\mathrm{Tr}[\gamma^+\Phi(\vec{S}_T)] - \mathrm{Tr}[\gamma^+\Phi](-\vec{S}_T)\Big] \propto f_{1T}^{\perp,(1)}(x)$$

Manipulation of Gauge Links + Impact parameter representation

$$\langle k_T^i \rangle(x) = \int d^2b_T \int \frac{dz^-}{2(2\pi)}\, \mathrm{e}^{ixP^+z^-} \langle P^+;\vec{0}_T;S_T|\bar{\psi}(z_1)\gamma^+[z_1;z_2]I^i(z_2)\psi(z_2)|P^+;\vec{0}_T;S_T\rangle$$

$$z_{1/2} = \mp\frac{z^-}{2}n_- + b_T$$

Impact parameter representation for GPD E

$$I^i(z^-) = \int dy^-\,[z^-;y^-]gF^{+i}(y^-)[y^-;z^-]$$ **coll. "soft gluon pole" matrix element**

- **(Semi-) classical approximation:** $\hat{I}^i \simeq \mathcal{I}^i(x, \vec{b}_T)\mathbf{1} + \ldots$

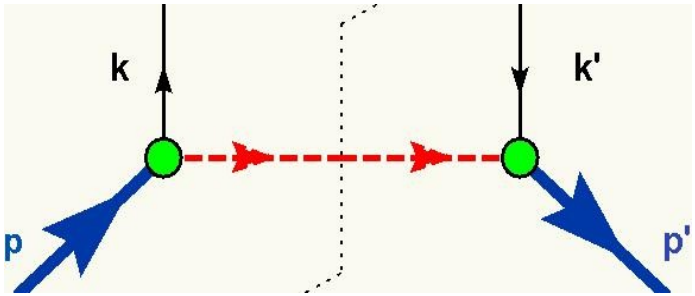
  $\rightarrow$ **factorization of final state interactions and spatial distortion:**

$$\langle k_T^i \rangle = -M \epsilon_T^{ij} S_T^j \, f_{1T}^{\perp,(1)}(x) \simeq \int d^2 b_T \, \mathcal{I}^i(x, \vec{b}_T) \, \frac{\vec{b}_T \times \vec{S}_T}{M} \, \frac{\partial}{\partial b_T^2} \mathcal{E}(x, \vec{b}_T^2)$$

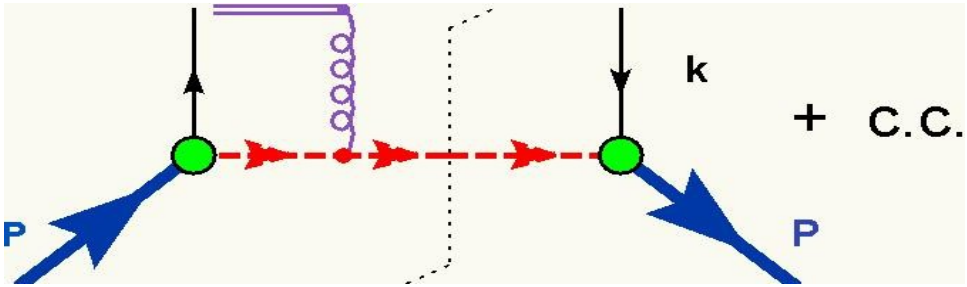
$\mathcal{I}^i(x, \vec{b}_T^2)$ **: Lensing Function = net transverse momentum**

- **Also for Boer-Mulders function and chirally-odd GPDs...**
- **Relation valid for lowest order diagrams in simple spectator models:**
  [Burkardt, Hwang, PRD69, 074032], [Meissner, Metz, Goeke, PRD76, 034002]

GPDs:



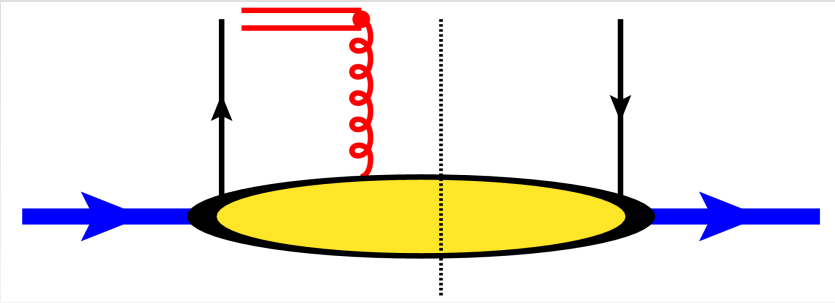
(T-odd) TMDs:



- **Analysis of *mother functions*: No exact, model-independent relations possible.**
  [Meissner, Metz, MS, Goeke, JHEP08(2008)038]
- ***"How approximate"* is the relation? Need to know the lensing function better...**

FSI beyond One-Gluon exchange

- **Final State Interactions (Sivers-effect) mostly modeled by a One-Gluon Exchange:**



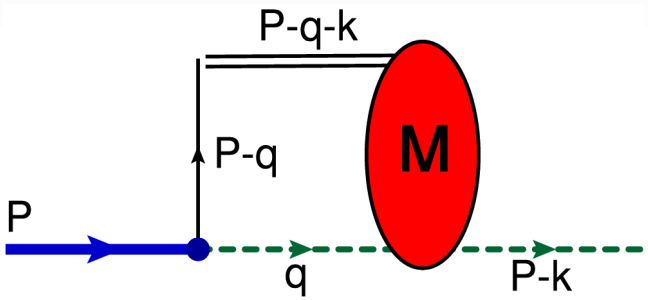
Example Diquark-spectator:

Sivers function $\propto \alpha_s$

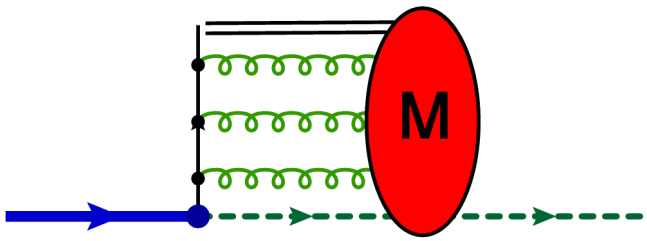
Sivers effect at HERMES ≈ 5%

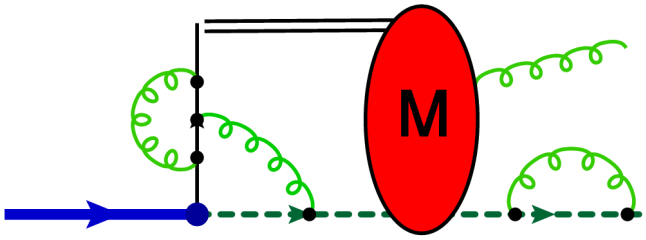
⇨ $\alpha_s \approx 0.2 - 0.3$

- **Relativistic Eikonal models: Treat FSI non-perturbatively.**

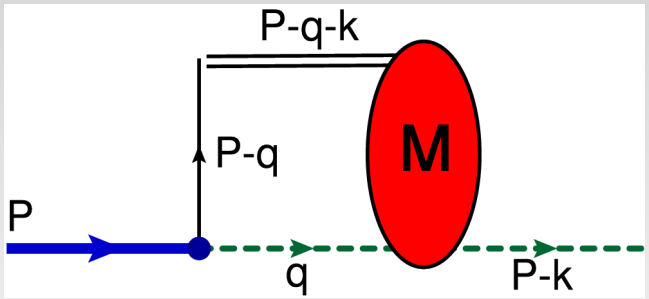


- **Work in the diquark picture (Lens. funct. identifyable) → more realistic for a pion...**
- **Only diagrams that reflect the "naive picture".**





Lensing Function



Assume a non-perturbative scattering amplitude M +

Separate GPD and FSI via contour integration

$$\int \frac{d^4q}{(2\pi)^4}\, g_N[(P-q)^2] \frac{[(\not P - \not q + m_q)u(P,S)]_i \mathcal{M}^{ab}_{bc}(q, P-k)}{[n\cdot(P-k-q)+i\varepsilon][(P-q)^2+m_q^2+i\varepsilon][q^2-m_s^2+i\varepsilon]}$$

- **Step 1: Integration over q^-:**
  Assume no q^- & q^+ poles in M.
  q^- - poles at one loop for higher twist T-odd TMDs [Gamberg, Hwang, Metz, MS, PLB 639, 508]
  $\rightarrow$ non-light like Wilson lines, would spoil relation.

- **Step 2: Integration over q^+:**
  Fix the q+ - pole
  emphasizes a "natural" picture of FSI
  equivalent to Cutkosky cut, assumptions of Step 1 valid in Eikonal models

$$\frac{1}{(1-x)P^+ - q^+ + i\varepsilon} = \mathrm{P}\frac{1}{(1-x)P^+ - q^+} - i\pi\delta((1-x)P^+ - q^+)$$

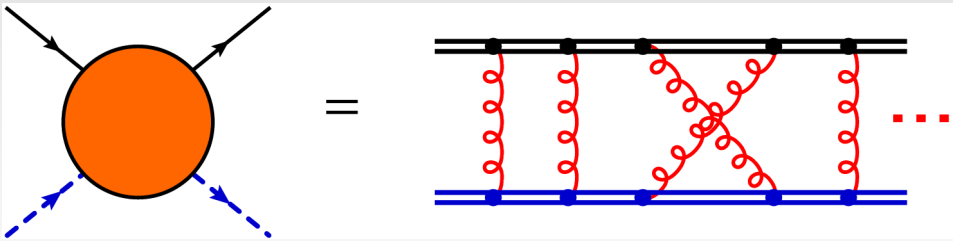
$$I^i(x,\vec{q}_T) = \int \frac{d^2p_T}{(2\pi)^2}(2p_T - q_T)^i \Im M^{ab}_{bc}(|\vec{p}_T|)\Big((2\pi)^2\delta^{ac}\delta^{(}2)(\vec{p}_T - \vec{q}_T) + \Re M^{cd}_{da}(|p_T - q_T|)\Big)$$

Relativistic Eikonal Models

Scattering amplitude of two highly energetic particles at small momentum transfer:

Possible under two assumptions:

1) Particles are eikonalized (not even an appr. for quark...) **2) Gener. Ladder Approx.**



In QED: Coulomb phase (70s)
1) Diagrammatic: Wu, Yang,...
2) Combinatoric: Levy, Sucher, Quiros...
3) Functional: Fried, Gabellini,...

Here: Outline of the functional approach for quark-antiquark scattering (pion):
[Fried, Gabellini, Avan, Eur.Phys.J. C13, 699]

QCD-generating functional:

$$Z[j,\eta,\bar\eta,\xi,\bar\xi] = \mathcal{N}\int \mathcal{D}A\, \mathrm{e}^{-\frac{i}{4}\int F^2 - \frac{i}{2}\int(\partial\cdot A)^2 + i\int j\cdot A}\, \mathrm{e}^{\mathrm{Tr}\ln G^{-1}[A] + \mathrm{Tr}\ln H^{-1}[A]}\, \mathrm{e}^{-i\int\bar\eta G[A]\eta - i\int\bar\xi H[A]\xi}$$

Quark-Antiquark 4-point function:

$$T(i_1,i_2|f_1,f_2) = \frac{\delta}{\delta\bar\eta(f_1)}\frac{\delta}{\delta\eta(i_1)}\frac{\delta}{\delta\bar\eta(i_2)}\frac{\delta}{\delta\eta(f_2)} Z\Big|_{j=\eta=\bar\eta=\xi=\bar\xi=0}$$

$$= -\mathcal{N}\int \mathcal{D}A\, \mathrm{e}^{-\frac{i}{4}\int F^2 - \frac{i}{2}\int(\partial\cdot A)^2}\, \mathrm{e}^{\mathrm{Tr}\ln G^{-1}[A] + \mathrm{Tr}\ln H^{-1}[A]}\, G(i_2,f_2|A)G(f_1,i_1|A)$$

Eikonal Approximation

Idea: highly energetic particle looses spin information $\gamma^\mu \longrightarrow v^\mu$

In QED:

$$G(x,y|A) = (i\partial\!\!\!/ - m_q + A\!\!\!/(x))^{-1}\delta^{(4)}(x-y)$$

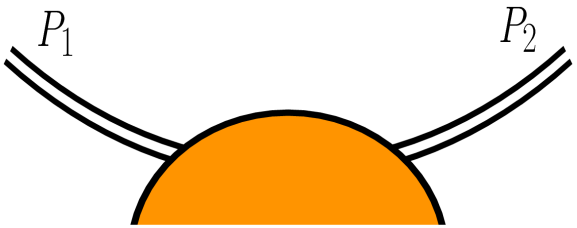
$$\longrightarrow (v\cdot(i\partial + A(x)) - m_q)^{-1}\delta^{(4)}(x-y) = -i\int_0^\infty ds\, \mathrm{e}^{is(v\cdot(\partial+A(x))-m_q+i\epsilon)}\delta^{(4)}(x-y)$$

$$\boxed{G^{ab}_{\mathrm{eik}}(x,y|A) = -i\int_0^\infty ds\, \mathrm{e}^{-is(m_q-i0)}\delta^{(4)}(x-y-sv)\left(\mathrm{e}^{-ig\int_0^s d\beta\, v\cdot A^\alpha(y+\beta v)\, t^\alpha}\right)^{ab}_{+}}$$

Trick to disentangle the A-field and the color matrices t: Functional FT

$$\left(\mathrm{e}^{-ig\int_0^s d\beta\, v\cdot A^\alpha(y+\beta v)\, t^\alpha}\right)^{ab}_{+} = \mathcal{N}'\int \mathcal{D}\alpha \int \mathcal{D}u\, \mathrm{e}^{i\int d\tau\, \alpha^\beta(\tau)u^\beta(\tau)}\mathrm{e}^{ig\int d\tau\, \alpha^\beta(\tau)\, v\cdot A^\beta(y+\tau v)}\left(\mathrm{e}^{i\int_0^s d\tau\, t^\beta u^\beta(\tau)}\right)^{ab}_{+}$$

Transformation of the scattering amplitude into momentum space + Amputation + Insertion of eikonal propagators:



$$\left(\frac{\partial}{\partial g}G^{\mathrm{eik}}\right)_{\mathrm{amp}}(P_2,P_1)$$

Eikonal Amplitude

$$\frac{\partial^2 M}{\partial g_1 \partial g_2} \simeq \int \mathcal{D}A\, \mathrm{e}^{i\int[-\frac{1}{4}F^2-\frac{\lambda}{2}(\partial\cdot A)^2)]+\mathrm{Tr}\ln G^{-1}[A]+\mathrm{Tr}\ln H^{-1}[A]} \left(\frac{\partial G}{\partial g_1}\right)\left(\frac{\partial \bar{G}}{\partial g_2}\right)$$

Further steps (in words) [Fried, Gabellini, Avan, Eur.Phys.J. C13, 699]:

- **Use scale invariance of the gauge vector n: "Quasi-Abelian limit"**
- **Separate off quadratic gluon terms**
- **Neglect Fermion loops and gluon self-interactions (Ladder approximation)**

Final result for the eikonal quark-antiquark scattering amplitude:

$$M^{\mathrm{eik}}_{ab}(q_T) = \frac{2(1-x)P^+}{m_s}\int d^2 z_T\, \mathrm{e}^{-iq_T\cdot z_T}\left[\int d^{N^2-1}\alpha \int \frac{d^{N^2-1}u}{(2\pi)^{N^2-1}}\left(\mathrm{e}^{i\chi(|z_T|)t\cdot\alpha}\right)_{ac}\left(\mathrm{e}^{it\cdot u}\right)_{cb}\mathrm{e}^{-i\alpha\cdot u}-1\right]$$

Abelian U(1)-theory: recover Coulomb phase $\ldots\left[\mathrm{e}^{i\chi(|z_T|)}-1\right]$

SU(2): analytical evaluation of color effects: $\mathrm{e}^{i\frac{\sigma}{2}\cdot u} = \cos\left(\frac{|u|}{2}\right) + i\frac{\sigma\cdot u}{|u|}\sin\left(\frac{|u|}{2}\right)$

SU(3): (?) numerical evaluation, in process...

The Eikonal Phase

Recall: $\chi(|z_T|) = g^2 \int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\infty} d\tau_2\, n^\mu \bar{n}^\nu \mathcal{D}_{\mu\nu}(z + \tau_1 n - \tau_2 \bar{n}) = \frac{\alpha}{\pi} \int d^2 k_T n^\mu \bar{n}^\nu \tilde{\mathcal{D}}_{\mu\nu}(k_T) e^{i\vec{k}_T \cdot \vec{z}_T}$

For a free gluon propagator: $\chi(|z_T|) = 2\alpha K_0(\mu |z_T|)$

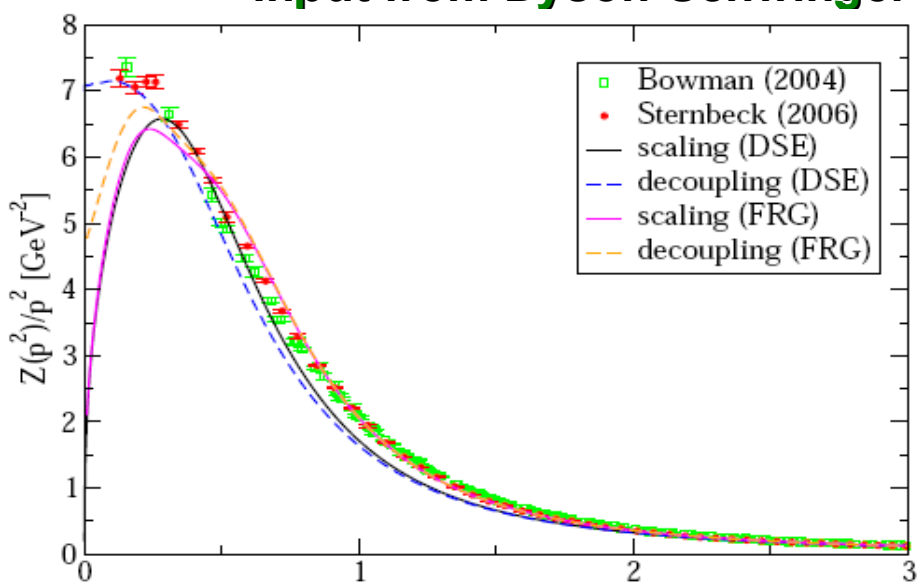
- **Good news:** **Recover exactly old perturbative results in the one gluon exchange.**
  [Burkardt,Hwang; Meißner, Metz, Goeke]

e.g.: scalar diquark: $\mathcal{I}^i(x, \vec{b}_T) = -\alpha(1-x)\frac{b_T^i}{\vec{b}_T^2}$ $\quad f_{1T}^{\perp}(x, \vec{k}_T^2) = -\frac{g^2}{2(2\pi)^3}\alpha \frac{(1-x)M(xM+m_q)}{\vec{k}_T^2(\vec{k}_T^2+\tilde{m}^2)} \ln\left(\frac{\vec{k}_T^2+\tilde{m}^2}{\tilde{m}^2}\right)$

- **Bad news:** **Dependence on IR-regulator.**

Exchanged gluons are soft → free propagator? What is the value of α_s in the IR-limit?

→ **Input from Dyson-Schwinger Equations.**



[Plot from Fischer, Maas, Pawlowski, arXiv:0810.1987]

- Calculations in Landau gauge + Euclidean space ⟶ applicable here.
- IR-finite gluon propagator $\mathcal{D}(p^2 \to 0) \sim (p^2)^{(2\kappa-1)}$
- Renorm. Point at $\mu_R = m_Z$ → here: μ_R = 1GeV
- DSE: IR-limit of $\alpha_s(p^2 \to 0) \sim 2.972$
- Use running of as as vertex factor.

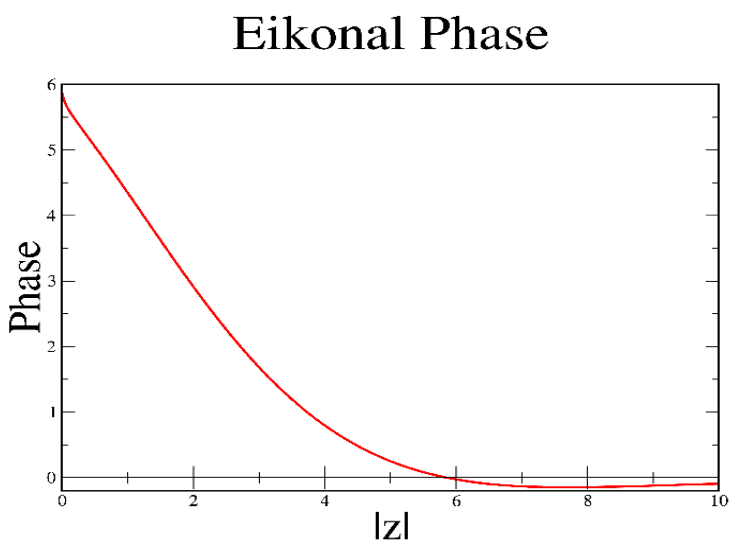
Lensing Function

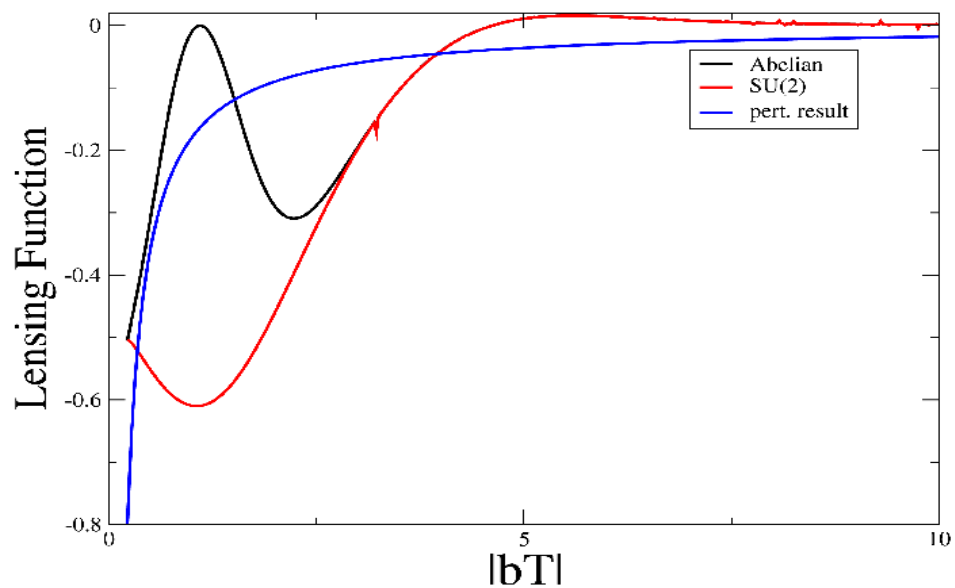
Express Lensing Function in terms of Eikonal Phase:

$$\mathcal{I}^i_{(N=1)}(x,\vec{b}_T) = \frac{1}{4}\frac{b^i_T}{|\vec{b}_T|}\chi'\left(\frac{|\vec{b}_T|}{1-x}\right)\left[1+\cos\chi\left(\frac{|\vec{b}_T|}{1-x}\right)\right]$$

$$\mathcal{I}^i_{(N=2)}(x,\vec{b}_T) = \frac{1}{4}\frac{b^i_T}{|\vec{b}_T|}\chi'\left(\frac{|\vec{b}_T|}{1-x}\right)\left[1+\cos\frac{\chi}{4}+\frac{\chi}{8}\left(\frac{\chi}{4}-\sin\frac{\chi}{4}\right)\right]\left(\frac{|\vec{b}_T|}{1-x}\right)$$

$$\mathcal{I}^i_{(N=3)}(x,\vec{b}_T) = ???$$

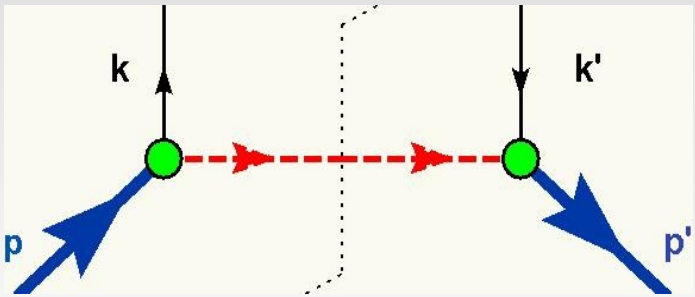




Final State Interactions remain negative (even in SU(2))

The GPD E in the diquark model

The remaining piece of the puzzle: GPD E



Scalar diquark with a multi-pole diquark vertex factor

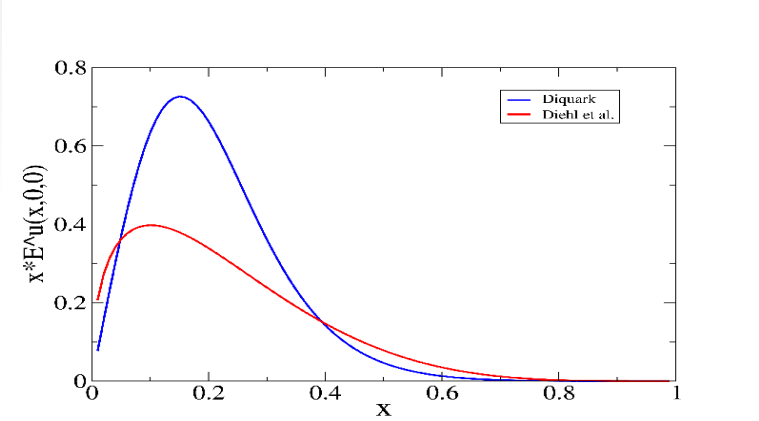
$$\Lambda^{2(\alpha-1)} \frac{k^2 - m_q^2}{[k^2 - \Lambda^2]^\alpha}$$

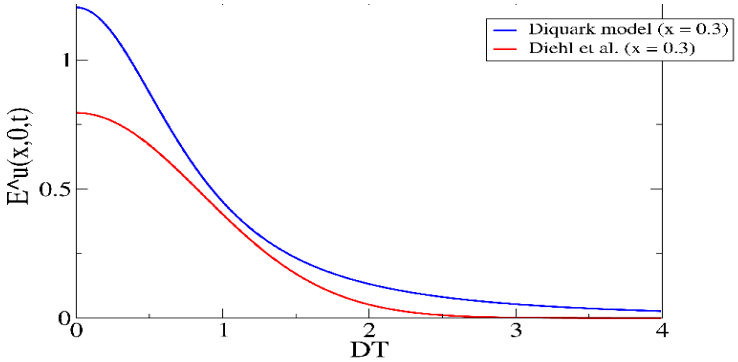
- *Need to fit the model parameters:*
  GPD-limits:
  Form Factors F1, F2 → t-dependence
  \+ valence u(x) (GRV) → x-dependence

- Difficult to fit all limits for diquark only
  → Form Factor fit preferred.

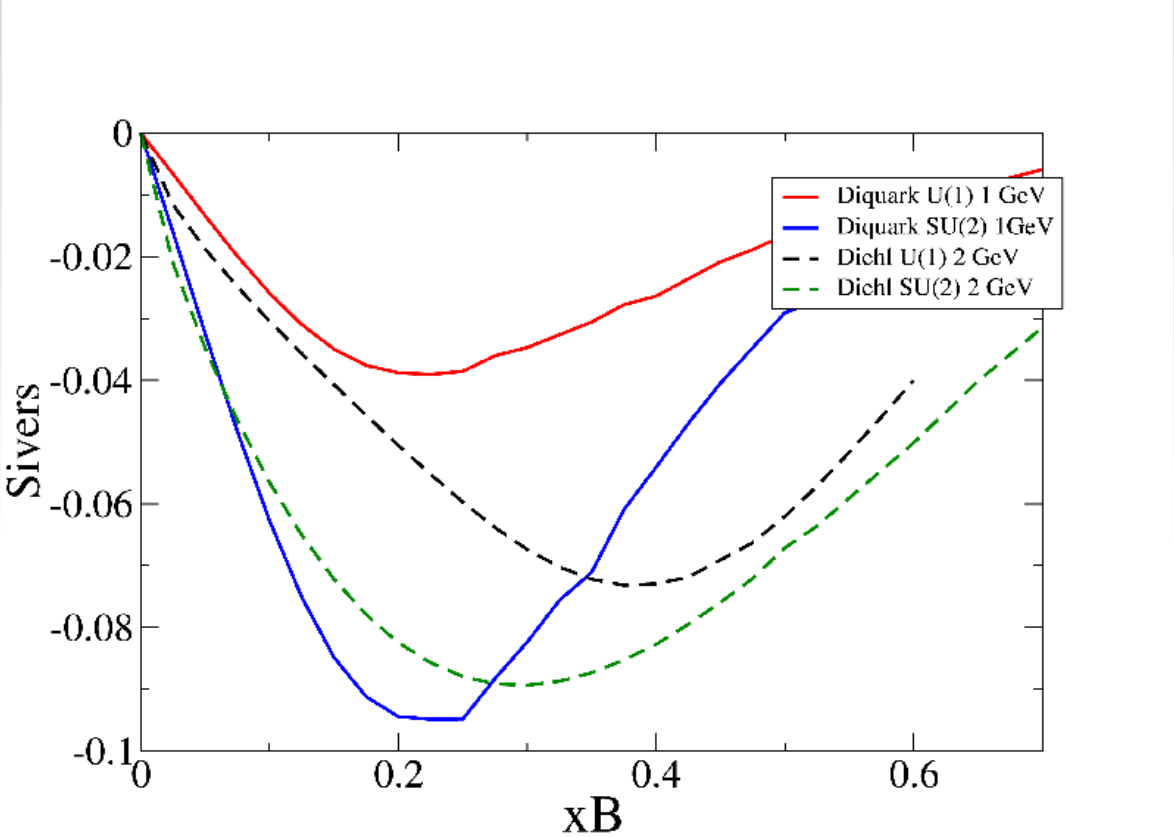
- Other GPD model:
  [Diehl et al., EPJ C39 (2005) 1-39]

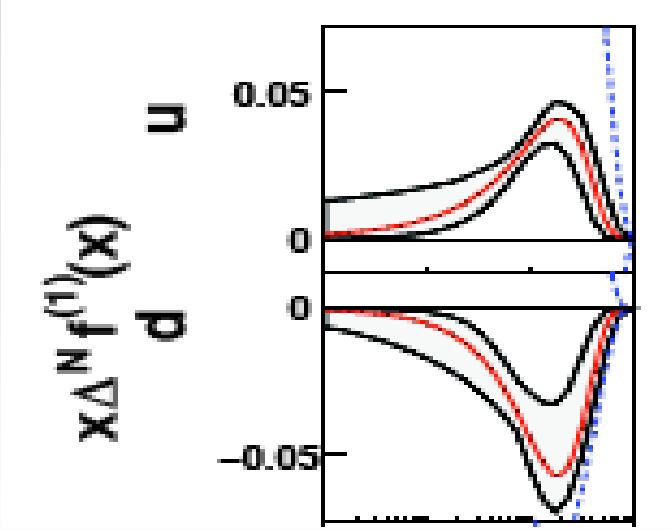
$$E_v^u(x,0,t) = (N^u \kappa^u x^{-\alpha}(1-x)^\beta) e^{t g^u(x)}$$





Results for the u-quark Sivers function





Extraction by Torino-Group

$$\Delta^N f^{(1)}(x) = -f_{1T}^{\perp}(x)$$

- **Relation produces a Sivers effect ~ 0.04 – 0.08, Torino - Extraction ~ 0.05.**
- **SU(2) color factors seem to increase the effect. Effect of SU(3) ???**
- **Relation overshoots extraction, but correct sign and order of magnitude...**

Summary:

- Relation Sivers – E via separation of FSI + spatial distortion of parton dist.

- Relation is not rigorous, model-dependent.
  Holds for lowest order spectator models.

- Relativistic Eikonal model: Non-perturbative, field-theoretical model of FSI
  → Lensing Function.

- Relation reproduces the right order of magnitude of the Sivers effect.

Outlook:

- Apply formalism for other soft objects:
  Boer-Mulders, Collins function, soft factor...

- Improve the diquark model through axial-vectors.

- SU(3) color factors

- Other Dyson-Schwinger extractions of Gluon-Prop. / running coupling.

The Quasi-Abelian Limit

$$\left(\frac{\partial}{\partial g}G^{\rm eik}\right)^{ab}_{\rm amp}(P_2,P_1)$$

$$=\mathcal{N}'\int d^4z\,{\rm e}^{-iz\cdot(P_1-P_2)}\int\mathcal{D}\alpha\int\mathcal{D}u\,\alpha^\beta(0)v\cdot A^\beta(z)\,{\rm e}^{ig\int_{-\infty}^{\infty}d\tau\,\alpha^\beta(\tau)v\cdot A^\beta(z+\tau v)}\,{\rm e}^{i\int\alpha\cdot u}\left({\rm e}^{i\int_{-\infty}^{\infty}d\tau\,t^\beta u^\beta(\tau)}\right)^{ab}_{+}$$

Description of the gauge link: Invariance under rescaling of gauge vector: $v=n\longrightarrow\lambda n$

$$\boxed{\int_{-\infty}^{\infty}\lambda d\tau\,\alpha^\beta(\tau)n\cdot A^\beta(z+\tau\lambda n)=\int_{-\infty}^{\infty}d\tilde\tau\,\alpha^\beta(\tfrac{\tilde\tau}{\lambda})n\cdot A^\beta(z+\tilde\tau n)\simeq\alpha^\beta(0)\int_{-\infty}^{\infty}d\tau\,n\cdot A^\beta(z+\tau n)}$$

"Quasi-Abelian Limit" [Fried, Gabellini, PRD 55,2430]

For spectator eikonal propagator: $\lambda\sim\dfrac{(1-x)P^+}{m_s}$ **"Infinite momentum frame"**

Back to the eikonal amplitude: Insertion of the quasi-abelian limit.

$$\boxed{\frac{\partial^2 M}{\partial g_1\partial g_2}\simeq\int\mathcal{D}A\,{\rm e}^{i\int[-\frac{1}{4}F^2-\frac{\lambda}{2}(\partial\cdot A)^2)]+{\rm Tr}\ln G^{-1}[A]+{\rm Tr}\ln H^{-1}[A]}\left(\frac{\partial G}{\partial g_1}\right)\left(\frac{\partial\bar G}{\partial g_2}\right)}$$

Eikonal Amplitude

Focus on the gluonic part: Separate off quadratic gluon terms

$$\frac{\partial^2 M}{\partial g_1 \partial g_2} \simeq \int \mathcal{D}A\, \mathrm{e}^{i\int[-\frac{1}{4}F^2-\frac{\lambda}{2}(\partial\cdot A)^2)]+\mathrm{Tr}\ln G^{-1}[A]+\mathrm{Tr}\ln H^{-1}[A]} \left(\frac{\partial G}{\partial g_1}\right)\left(\frac{\partial \bar{G}}{\partial g_2}\right)$$

Write it in terms of a Gaussian integral:

$$\propto \ldots \left[\int \mathcal{D}A\, \mathrm{e}^{-\frac{i}{2}\int A\mathcal{D}^{-1}A+i\int J\cdot A}\mathcal{F}[A]\right]$$

$$\mathcal{F}[A] = \mathrm{e}^{-\frac{i}{4}\int(F^2-f^2)}\mathrm{e}^{\mathrm{Tr}\ln G^{-1}[A]+\mathrm{Tr}\ln H^{-1}[A]} = \int \mathcal{D}\phi\, \tilde{\mathcal{F}}[\phi]\mathrm{e}^{i\int A\cdot\phi}$$

$\mathcal{D}^{-1}$: **inverse free gluon propagator**

Eikonal current in the quasi-abelian limit:

$$J^{\mu,\beta}(x) = g_1\alpha^\beta(0)\int_{-\infty}^{\infty} d\tau\, n^\mu \delta^{(4)}(x-z-\tau n) + g_2\bar{\alpha}^\beta(0)\int_{-\infty}^{\infty} d\tau\, \bar{n}^\mu \delta^{(4)}(x-\bar{z}+\tau\bar{n})$$

Manipulation via functional translation: $\mathcal{F}[A+B] = \mathrm{e}^{\int G\frac{\delta}{\delta A}}\mathcal{F}[A]$

$$\int \mathcal{D}\phi\, \mathrm{e}^{i\int(J+\phi)\mathcal{D}(J+\phi)}\tilde{\mathcal{F}}[\phi] = \mathrm{e}^{\frac{i}{2}\int J\mathcal{D}J}\mathrm{e}^{-\frac{i}{2}\int\frac{\delta}{\delta A}\mathcal{D}\frac{\delta}{\delta A}}\mathcal{F}[A+\int\mathcal{D}J]\Big|_{A\to 0}$$

Eikonal Phase (gauge invariant for light-like "velocity" vectors):

$$\int J\mathcal{D}J = 2\alpha(0)\cdot\hat{\alpha}(0)\chi(z_T-\bar{z}_T) = 2\alpha(0)\cdot\hat{\alpha}(0)\int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\infty} d\tau_2\, n^\mu \bar{n}^\nu \mathcal{D}_{\mu\nu}(z-\bar{z}+\tau_1 n-\tau_2\bar{n})$$

Collecting all parts:

$$\int \mathcal{D}\alpha\mathcal{D}u \int \mathcal{D}\bar{\alpha}\mathcal{D}\bar{u} e^{i\int(\bar{u}\cdot\bar{\alpha}+u\cdot\alpha)}\left(e^{i\int t\cdot u}\right)_+\left(e^{i\int t\cdot\bar{u}}\right)_+ e^{i\alpha(0)\cdot\bar{\alpha}(0)\chi(z_T)} e^{-\frac{i}{2}\int\frac{\delta}{\delta A}\mathcal{D}\frac{\delta}{\delta A}}\mathcal{F}\left[A+\int\mathcal{D}J\right]\Big|_{A\to 0}$$

- **Quasi-Abelian limit** → **path-integrals reduce to 1 for $\tau \neq 0$ → ordinary integrals**
- **Ladder Approximation: Neglect fermion loops and gluon-self interactions: $\mathcal{F} \to 1$**

Final result for the eikonal quark-antiquark scattering amplitude:

$$M_{ab}^{\text{eik}}(q_T) = \frac{2(1-x)P^+}{m_s}\int d^2 z_T\, e^{-iq_T\cdot z_T}\left[\int d^{N^2-1}\alpha \int \frac{d^{N^2-1}u}{(2\pi)^{N^2-1}}\left(e^{i\chi(|z_T|)t\cdot\alpha}\right)_{ac}\left(e^{it\cdot u}\right)_{cb} e^{-i\alpha\cdot u} - 1\right]$$

Abelian U(1)-theory: recover Coulomb phase $\ldots\left[e^{i\chi(|z_T|)}-1\right]$

SU(2): analytical evaluation of color effects: $e^{i\frac{\sigma}{2}\cdot u} = \cos\left(\frac{|u|}{2}\right) + i\frac{\sigma\cdot u}{|u|}\sin\left(\frac{|u|}{2}\right)$

SU(3): (?) numerical evaluation, in process...