C207 problem set 1 solutions

a): Using

$$L = 4\pi R^2 \sigma_{\rm sb} T^4 \tag{1}$$

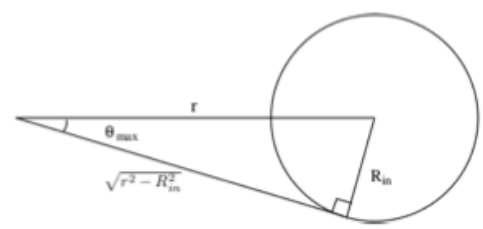
and solving for R gives, $R \approx 10^{15}$ cm. This is about 10,000 times the radius of the sun, and about 100 times larger than a red giant. The surface temperatures of young supernovae are not much hotter than the sun, but the supernova is much brighter because of the huge emitting surface area.

b: We consider the mean intensity from a sphere emitting uniform intensity I_0. At some point a distance r away from the source we see from the geometry that intensity is non-zero only between some angles $(0, \theta_{max})$, where θ_{max} is defined by finding the ray that is just tangent to the sphere. This gives

$$\begin{aligned} J &= \frac{1}{4\pi}\int_0^{\theta_{max}}\int_0^{2\pi} I_0\, d\phi\, \sin\theta\, d\theta \\ &= \frac{2\pi}{4\pi} I_0(-\cos\theta)\Big|_0^{\theta_{max}} \\ &= \frac{1}{2} I_0\,(1-\cos\theta_{max}) \end{aligned}$$

But

$$\begin{aligned} \cos\theta_{max} &= \frac{\sqrt{r^2 - R_{in}^2}}{r} \\ &= \left[1 - \frac{R_{in}^2}{r^2}\right]^{1/2} \end{aligned}$$



This gives

$$J_\nu = W(r) I_0 \quad \text{where } W(r) = \frac{1}{2}[1 - \sqrt{1 - R^2_{\rm sn/r^2}}] \tag{2}$$

At $r = R_{\rm sn}$ we have $J_\nu = 1/2 I_0$, which makes sense since someone sitting at the surface of the sphere sees an intensity I_0 coming from one hemisphere, but no intensity from the other. For $r \gg R_{\rm sn}$ we expand the square root and find $J_\nu = I_0 R_{\rm sn}^2/4r^2$. This makes sense because at this distance, the solid area that the source covered can be approximated as just its cross-sectional area ($\pi R_{\rm sn}^2$) divided by the complete sphere of that distance ($4\pi r^2$).

c: Taking $I_0 = B_\nu(T_{\rm sn})$ and integrating over all frequencies gives

$$J = W(r) B(T_{\rm sn}) = W(r)\frac{\sigma_{\rm sb} T_{\rm sn}^4}{\pi} \tag{3}$$

Since the supernova luminosity is $L = 4\pi\sigma_{\rm sb}T_{\rm sn}^4 R_{\rm sn}^2$ this becomes

$$J = W(r)\frac{L}{4\pi^2 R_{sn}^2} \tag{4}$$

In the limit $r \gg R_{\rm sn}$ this becomes

$$J = \frac{L}{16\pi^2 r^2} = \frac{1}{4\pi}\frac{L}{4\pi r^2} \tag{5}$$

This is not too surprising, since at large r the radiation is completely radial directed and so the zeroth and first moments of the radiation field are equal. Thus $J = F/4\pi$ where F is the flux.

d: The dust cloud is at a position (x, z) relative to the supernova. Rest et. al 2008 measured a separation distance of 4.6 degrees which is ≈ 0.08 radians. Given the distance of $D = 3$ kpc, this means that the x distance is

$$x = D\tan^{-1}(0.1 \text{ radians}) \approx 240 \text{ pc} \tag{6}$$

Note that Rest et al adopt a distance of $D = 2.3$ kpc, in which case $x \approx 184$ pc.

The distance between the supernova and cloud is $r_e = (x^2 + z^2)^{1/2}$. So we also need to figure out z from the light travel time. The echoed light has to travel a distance r_e to get to the cloud, and then another distance z to get back to the $z = 0$ point where the original supernova light was emitted. The distance the light has to travel is thus

$$l = r_e + z = (x^2 + z^2)^{1/2} + z \tag{7}$$

In this convention, z increases away from the observer. Note that Rest et al use the opposite sign convention. At a given time, t, we may see echoes from the surface corresponding to a given light travel distance, $l = ct$. Plugging this in and moving z over to the left hand side

$$ct - z = (x^2 + z^2)^{1/2} \tag{8}$$

Now squaring both sides

$$c^2t^2 + z^2 - 2ctz = x^2 + z^2 \tag{9}$$

Finally, solving for z gives

$$z = -\frac{x^2}{2ct} + \frac{ct}{2} \tag{10}$$

which is an equation of a paraboloid. Given that we know the time between Tycho's supernova and Rest's paper is $t \approx 436$ years, and we know the value of x from the image, we can solve find $z = 126$ pc. Then we get the distance as $r_e = 271$ pc. These values are slightly different if one uses $D = 2.3$ instead of 3.0 kpc.

e: Since the emission from the cloud is assumed to be due to isotropic scattering, the source function is simply $S_\nu = J_\nu(r_c)$. We assume the source function is constant over the cloud, in which case the solution to the transfer equation is

$$I_\nu = I_{\rm bg}e^{-\tau} + S_\nu(1 - e^{-\tau}) \tag{11}$$

In this case, there is no background source ($I_{\rm bg} = 0$) and we are in the optically thin limit, so expanding in τ gives

$$I_\nu = S_\nu\tau \tag{12}$$

A useful expression for optically thin sources. For a uniform density medium, the optical depth along a ray with impact parameter p is simply given by $\tau = \alpha s(p)$ where $s(p)$ is the path length

through the cloud along that ray (see Figure 1 of the problem set). From the geometry, we have $s(p) = 2[R_c^2 - p^2]^{1/2}$ so that the brightness profile is

$$I_\nu(p) = 2J_\nu \alpha R_c [R_c^2 - p^2]^{1/2} = 2J_\nu \tau_0 [1 - (p/R_c)^2]^{1/2} \tag{13}$$

where $\tau_0 = \alpha R_c$ is the optical depth of the cloud radial from the center to the surface.

f: To get the flux, we need to integrate the specific intensity of eq. 13) over all angles.

$$F_\nu = \oint \mu I_\nu(\mu) d\mu d\phi = 2\pi \int I_\nu d\mu \tag{14}$$

Here μ is the angle the observer is looking to see intensity at an impact parameter p. From the geometry,

$$\mu = \cos\theta = \frac{D}{(p^2 + D^2)^{1/2}} \approx 1 - \frac{1}{2} p^2/D^2 \tag{15}$$

Where the expansion holds since $D \gg p$. We can convert the integral over μ into one over p by noting

$$d\mu \approx \frac{pdp}{D^2} \tag{16}$$

working to leading order in p/D we can take $\mu \approx 1$ and the integral becomes

$$F_\nu = \frac{2\pi}{D^2} \int_0^{R_c} I_\nu(p) p dp \tag{17}$$

Plugging in our solution for $I_\nu(p)$ and integrating we find

$$F_\nu = \frac{4\pi}{3} \tau_0 (R_c/D)^2 J_\nu \tag{18}$$

g: The emission coefficient is defined as $j_\nu = \alpha S_\nu$. It gives the power emitted per unit volume, per frequency bin, per steradian. If we integrate over all directions and over the cloud volume, we therefore get the monochromatic luminosity (i.e., power per frequency bin)

$$L_\nu = \oint \int j_\nu dV d\Omega = 4\pi j_\nu (4\pi R_c^3/3) \tag{19}$$

where we used the fact that the emission is isotropic to get the factor of 4π and the fact that j_ν is constant in space to get the volume factor. This then gives

$$L_\nu = \frac{16\pi^2}{3} \alpha J_\nu R_c^3 \tag{20}$$

and dividing by $4\pi D^2$ to get the observed flux gives

$$F_\nu = \frac{4\pi}{3} \tau_0 J_\nu (R_c/D)^2 \tag{21}$$

which is the same as that found in the previous part of the problem.

h: In the limit, $r_e \gg R_{\rm sn}$ the source function of the cloud is

$$S_\nu = J_\nu = B_\nu(T_{\rm sn}) \frac{R_{\rm sn}^2}{4r_e^2} \tag{22}$$

Thus the observed flux from the echo is, using eq. 21

$$F_{\nu,\mathrm{echo}} = \frac{4\pi\tau_0}{3} B_\nu(T_{\mathrm{sn}}) \frac{R_{\mathrm{sn}}^2}{4r_e^2} \frac{R_c^2}{D^2} = \frac{\tau_0}{3} [\pi B_\nu(T_{\mathrm{sn}})] \frac{R_{\mathrm{sn}}^2}{D^2} \frac{R_c^2}{r_e^2} \tag{23}$$

Meanwhile, the monochromatic luminosity of the supernova is

$$L_{\nu,\mathrm{sn}} = 4\pi[\pi B_\nu(T_{\mathrm{sn}})] R_{\mathrm{sn}}^2 \tag{24}$$

(Don't forget the factor of pi that comes in from the flux from a uniform intensity sphere. As a check, note that when we integrate B_ν over all wavelengths we get $\sigma_{\mathrm{sb}} T_{\mathrm{sn}}^4/\pi$ so this expression gives us the familiar luminosity of a blackbody sphere). The supernova flux is

$$F_{\nu,\mathrm{sn}} = \frac{L_{\nu,\mathrm{sn}}}{4\pi D^2} = \frac{\pi B_\nu(T_{\mathrm{sn}}) R_{\mathrm{sn}}^2}{D^2} \tag{25}$$

so the ratio is

$$\frac{F_{\nu,\mathrm{echo}}}{F_{\nu,\mathrm{sn}}} = \frac{\tau_0}{3} (R_{\mathrm{c}}/r_e)^2 \tag{26}$$

Plugging in values of $\tau_0 \approx 0.1$ and the radii, we find the ratio is of order 10^{-13}.