1 Health Benefits of Negative Hydrogen

a) Since we assume LTE, we can apply the Saha equation

$$\frac{n_p n_e}{n_{HI}} = 2\left[\frac{2\pi m_e kT}{h^2}\right]^{3/2} e^{-\chi/kT} = \Phi(T) \tag{1}$$

For pure hydrogen, we include the charge conservation equation, $n_e = n_p$. This system has an analytic solution

$$x_{\rm HII} = -\frac{\Phi}{2n} + \frac{1}{2n}[\Phi^2 + 4\Phi n]^{1/2} \tag{2}$$

where n is the density. Plugging in numbers gives $x_{\rm HII} \approx 8 \times 10^{-5}$. The optical depth to electron scattering is, using the Thompson cross section and the scale height, H, is

$$\tau = n x_{\rm HII} \sigma_t H \approx 5.4 \times 10^{-5} \tag{3}$$

b) The expression for the free-free extinction coefficient is available on the webpage, and depends on the product of n_p and n_e. The optical depth to free-free is

$$\tau = \alpha_{\rm ff} H \approx 1.5 \times 10^{-5} \tag{4}$$

c) The bound-free cross-section, in the hydrogenic approximation, can be estimated

$$\sigma_{\rm bf} \approx \frac{\sigma_{\rm tot}}{\nu_t}\left(\frac{\nu}{\nu_t}\right)^{-3} \tag{5}$$

where the threshold frequency is

$$\nu_t = \chi_i/h = \frac{\rm Rydberg}{hn^2} \tag{6}$$

The cross-section is zero for $\nu < \nu_t$. The threshold wavelength for $n = 1$ is 912 Å, for $n = 2$ is 3648 Å, and for $n = 3$, is 7296 Å. Therefore the $n = 1, 2$ levels do not contribute any bound-free opacity at a wavelength 5000 Å. For the $n = 3$ level, we need to determine the level population, which we can using the LTE Boltzmann factor

$$n_3 = \frac{g_3}{U(t)} e^{-\Delta E/kT} \tag{7}$$

where the LTE partition function is

$$U(T) = \sum_i g_i e^{-\Delta E_i/kT} \tag{8}$$

and the statistical weight is $g_i = 2n^2$ for hydrogen. Since the majority of the atoms are in the ground state, one can approximate $U(T) \approx g_1 = 2$, although it is not difficult to calculate the first few terms. Putting it all together, we find the optical depth to bound-free in the $n = 3$ state.

$$\tau = n_3 \sigma_{\rm bf} H \approx 3 \times 10^{-4} \tag{9}$$

d) We can treat the H^- fraction using the Saha equation in analogy to part a). The equation is then

$$\frac{n_{\rm HI} n_e}{n_{\rm H^-}} = 2\left[\frac{2\pi m_e kT}{h^2}\right]^{3/2} e^{-\chi^-/kT} = \Phi(T) \tag{10}$$

where $\chi^- = 0.75$ eV. To this we must add the particle conservation equation

$$n_{\rm HI} + n_p + n_{\rm H^-} = n \tag{11}$$

and the charge conservation equation

$$n_e = n_p - n_{\mathrm{H}^-} \tag{12}$$

These equations, along with the Saha expression Equation 1 form a complete system of equations. Solving this system of equations can get rather involved and is best tackled numerically. To simplify, however, let's make the ansatz that the density of of ionized hydrogen is much greater than that of negative hydrogen. In this case, the free electron density is what we found in part a). We can then easily solve the Saha equation to determine the number density of H-minus. Then, after the fact, we can shows that $n_{H^-} \ll n_{\mathrm{HII}}$, which confirms the ansatz we made to start.

The threshold frequency for H-minus bound-free is $\nu_t = 0.75\ \mathrm{eV}/h \approx 1.8 \times 10^{14}$ Hz. Plugging in numbers we see that $\sigma_{\mathrm{bf}} \approx 3 \times 10^{-18}$. Calculating the optical depth we find $\tau \sim 0.1$. Given that our hydrogenic cross-section is in lower than the actual value, this is not too bad.

2 Flipping Spins at the Epoch of Reionization

Define the excitation energy for the 21 cm line to be $T_\star = h\nu_{\mathrm{fs}}/k = 0.068$ K, where $\nu_{\mathrm{fs}} = (c/21\ \mathrm{cm})$. We are always in the Rayleigh-Jeans limit since $T \gg T_\star$ so the observed specific intensity is

$$I_{\nu,\mathrm{obs}} = \frac{2\nu_{fs}^2}{c^2} k T_b, \tag{13}$$

where T_b is the brightness temperature, which may or may not be related to the kinetic temperature of the observed gas, T_K. Lastly, the spin temperature T_s is defined as

$$\frac{n_1}{n_0} = \frac{g_1}{g_0} e^{-T_\star/T_s} = 3e^{-T_\star/T_s}. \tag{14}$$

a) Consider a CMB beam with specific intensity $B_\nu(T_\gamma, \nu_{fs}) \equiv I_0$ passing through a neutral hydrogen cloud of optical depth τ. The radiative transfer equation tells us that

$$\frac{\partial I_\nu}{\partial \tau} = -I_\nu + S_\nu. \tag{15}$$

If S_ν is constant in space, we can let $x = I_\nu - S_\nu$ and the solution to the above equation is

$$x = x_0 e^{-\tau}. \tag{16}$$

Substituting back in for x, we find that the observed specific intensity for the CMB beam is:

$$I_{\nu,\mathrm{obs}} = B_\nu(T_\gamma, \nu_{\mathrm{fs}}) e^{-\tau} + S_\nu(\nu_{\mathrm{fs}})(1 - e^{-\tau}). \tag{17}$$

Now, we must calculate the source function of the 21 cm line, $S_\nu = j_\nu/\alpha_\nu$. In terms of the Einstein coefficients, the extinction coefficient corrected for stimulated emission is

$$\alpha_\nu = \frac{h\nu_{\mathrm{fs}}}{4\pi} \phi(\nu_{\mathrm{fs}}) n_0 B_{01} \left(1 - \frac{n_1 B_{10}}{n_0 B_{01}}\right). \tag{18}$$

Now use the Einstein relation $B_{10}/B_{01} = g_0/g_1$ and substitute in for n_1/n_0:

$$\alpha_\nu = \frac{h\nu_{\mathrm{fs}}}{4\pi} \phi(\nu_{\mathrm{fs}}) n_0 B_{01} \left(1 - e^{-T_\star/T_s}\right). \tag{19}$$

Now use another Einstein relation $A_{10} = (2h\nu_{\mathrm{fs}}^3/c^2)B_{10} = (2h\nu_{\mathrm{fs}}^3/3c^2)B_{01}$,

$$\alpha_\nu = \frac{3c^2}{8\pi\nu_{\rm fs}^2}\phi(\nu_{\rm fs})n_0 A_{10}\left(1 - e^{-T_\star/T_s}\right). \tag{20}$$

Lastly, we can Taylor expand the exponential since $T_s \gg T_\star$ to obtain the extinction coefficient for the 21 cm line:

$$\alpha_\nu = \frac{3c^2}{8\pi\nu_{\rm fs}^2}\phi(\nu_{\rm fs})n_0 A_{10}\left(\frac{T_\star}{T_s}\right). \tag{21}$$

We can also express the emission coefficient in terms of the Einstein coefficients:

$$j_\nu = \frac{h\nu_{\rm fs}}{4\pi}\phi(\nu_{\rm fs})n_1 A_{10}. \tag{22}$$

Now we can compute the source function by dividing the emission and extinction coefficients:

$$S_\nu = \frac{2h\nu_{\rm fs}^3}{3c^2}\frac{n_1}{n_0}\frac{T_s}{T_\star} = \frac{2h\nu_{\rm fs}^3}{c^2}\frac{T_s}{T_\star}e^{-T_\star/T_s}. \tag{23}$$

In the limit of $T_s \gg T_\star$, we obtain the final result for the source function

$$S_\nu = \frac{2\nu_{\rm fs}^2}{c^2}kT_s. \tag{24}$$

Now return to equation 17 and plug in for the source function:

$$I_{\nu,\rm obs} = B_\nu(T_\gamma, \nu_{\rm fs})e^{-\tau} + \frac{2\nu_{\rm fs}^2}{c^2}kT_s(1 - e^{-\tau}). \tag{25}$$

Plugging in for $I_{\nu,\rm obs}$, we find the brightness temperature to be:

$$T_b = T_\gamma e^{-\tau} + T_s(1 - e^{-\tau}). \tag{26}$$

Relative to the CMB background temperature, we have $\delta T_b \equiv T_b - T_\gamma$, or

$$\boxed{\delta T_b = (T_s - T_\gamma)(1 - e^{-\tau})}. \tag{27}$$

b) In statistical equilibrium, the balance of Einstein coefficients is

$$n_1(A_{10} + B_{10}\bar{J} + C_{10}) = n_0(B_{01}\bar{J} + C_{01}), \tag{28}$$

or

$$\frac{n_1}{n_0} = \frac{B_{01}\bar{J} + C_{01}}{A_{10} + B_{10}\bar{J} + C_{10}}. \tag{29}$$

Use the following three Einstein relations:

$$\begin{aligned} B_{01} &= 3B_{10}, \\ A_{10} &= 2h\nu_{\rm fs}^3 B_{10}/c^2, \\ C_{01} &= 3C_{10}e^{-T_\star/T_K}. \end{aligned}$$

Also, the local mean intensity field $\bar{J}$ is defined such that

$$\bar{J} = \frac{2\nu_{\rm fs}^2}{c^2} kT_\gamma. \tag{30}$$

Using the above 4 relations, we can express equation 29 in terms of C_{10} and A_{10}. Defining $x_c = C_{10}T_\star/A_{10}T_\gamma$, we obtain the desired result:

$$T_s^{-1} = \frac{T_\gamma^{-1} + x_c T_K^{-1}}{1 + x_c}. \tag{31}$$

From inspection of the above equation we see that $\boxed{T_s = T_K \text{ when } x_c \gg 1}$ and $\boxed{T_s = T_\gamma \text{ when } x_c \ll 1}$. When $x_c \gg 1$, collisional de-excitation is dominant so the gas collisions are setting the spin-flip level populations and the spin temperature will approach the gas temperature. In the opposite limit, the level populations are being set by the radiative transitions caused by the CMB so the spin temperature approaches T_γ in this limit.

c) We want to find the critical density such that $x_c = 1$. For collisional de-excitation by neutral hydrogen collisions, the rate is roughly given by

$$C_{10} \simeq \sigma_{10} n_H v_H. \tag{32}$$

We will assume the cross section is just the geometric cross section, $\sigma_{10} = \pi a_0^2$, where a_0 is the Bohr radius. Assume a characteristic velocity of order

$$v_H \simeq \sqrt{\frac{kT_K}{m_H}}. \tag{33}$$

Plugging in we find the critical density to be:

$$n_{H,c} = \frac{A_{10}T_\gamma}{\pi a_0^2 T_\star}\sqrt{\frac{m_H}{kT_K}}. \tag{34}$$

In class we estimated the spontaneous emission coefficient for hyperfine splitting to be $A_{10} \simeq 6\times10^{-15}\ \mathrm{s}^{-1}$. Using $T_K = 100$ K and $T_\gamma = 2.7$ K, the critical density is roughly $\boxed{n_{H,c} \simeq 3\times10^{-2}\ \mathrm{cm}^{-3}}$.

d) The rate at which Lyα photons drive transitions from the excited to the ground hyperfine level is given by P_{10}. By LTE arguments (similar to the derivation for the relation between C_{10} and C_{01} from class), we have

$$P_{01} = 3P_{10}e^{-T_\star/T_K}. \tag{35}$$

Statistical equilibrium now including transitions due to Lyα photons is given by

$$n_1(A_{10} + B_{10}\bar{J} + C_{10} + P_{10}) = n_0(B_{01}\bar{J} + C_{01} + P_{01}). \tag{36}$$

Analogous to part b), we can use the Einstein relations and the relation for $\bar{J}$ to express the above equation only in terms of A_{10}, C_{10}, and P_{10}. The relation we obtain is

$$T_s^{-1} = \frac{T_\gamma^{-1} + T_K^{-1}(x_c + x_\alpha)}{1 + x_c + x_\alpha}, \tag{37}$$

where

$$x_\alpha = \frac{P_{10}}{A_{10}} \frac{T_\star}{T_\gamma}. \tag{38}$$

e) From part a), the observed fluctuation in the brightness temperature is given by equation 26. Absorption occurs when $\delta T_b < 0$ and emission occurs when $\delta T_b > 0$. From inspection of equation 26, the sign of δT_b depends on the quantity $T_s - T_\gamma$, so we must determine the sign of this quantity as a function of redshift.

(a) $200 \leq z \leq 1100$

- $n_H > n_{H,c}$ so we are in the limit of $x_c \gg 1$
- we know from part b) that $T_s = T_K$ in this limit
- the gas and CMB are still thermally coupled so $T_K = T_\gamma$
- so we have $T_s = T_\gamma$ and $\boxed{\delta T_b = 0}$ $\rightarrow$ no signal

(b) $40 \leq z \leq 200$

- $n_H > n_{H,c}$ still so we are in the limit of $x_c \gg 1$ and $T_s = T_K$
- the gas falls out of equilibrium with the CMB and cools adiabatically
- for a monotonic adiabatic gas, with no heating sources:

$$TV^{2/3} = constant. \tag{39}$$

The volume scales with redshift as $(1+z)^{-3}$ so we see that

$$T_K \propto (1+z)^2. \tag{40}$$

The CMB temperature scales with redshift as $T_\gamma \propto 1+z$. Thus, we see that once the gas falls out of equilibrium with the CMB, the gas will cool faster and $T_K < T_\gamma$.

- So we have:

$$T_s - T_\gamma = T_K - T_\gamma < 0, \tag{41}$$

and we see that $\boxed{\delta T_b < 0}$ $\rightarrow$ absorption.

(c) $30 \leq z \leq 40$

- now $n_H < n_{H,c}$ so we are in the limit of $x_c \ll 1$
- we know from part b) that $T_s = T_\gamma$ in this limit
- so we have $T_s - T_\gamma = T_\gamma - T_\gamma$ and $\boxed{\delta T_b = 0}$ $\rightarrow$ no signal

(d) $15 \leq z \leq 30$

- the production of Lyα photons leads to Lyα scattering setting the spin-flip level populations so $x_\alpha \gg 1$
- we know from part d) that $T_s = T_K$ in this limit
- due to adiabatic cooling and a lack of heating, $T_K < T_\gamma$
- so we have $T_s - T_\gamma = T_K - T_\gamma < 0$ and $\boxed{\delta T_b < 0}$ $\rightarrow$ absorption

(e) $7 \leq z \leq 15$

- the production of Lyα photons leads to Lyα scattering setting the spin-flip level populations so $x_\alpha \gg 1$
- we know from part d) that $T_s = T_K$ in this limit
- first sources heat the gas such that now $T_K > T_\gamma$

- so we have $T_s - T_\gamma = T_K - T_\gamma > 0$ and $\boxed{\delta T_b > 0}$ $\rightarrow$ emission

(f) $z \leq 7$

- reionization has ionized the universe so neutral hydrogen is negligible
- thus, $x_c = x_\alpha = 0$ and $T_s = T_\gamma$
- so we have $T_s - T_\gamma = T_\gamma - T_\gamma$ and $\boxed{\delta T_b = 0}$ $\rightarrow$ no signal