

# 1 The Rise of Stromgren Spheres

**1a)** Defining  $\dot{N}_{\text{ion}}$  ( $\dot{N}_{\text{rec}}$ ) as the total number of photoionizations (recombinations) per second that occur in the surrounding gas, we have the expression

$$\frac{dN_{\text{tot}}}{dt} = \dot{N}_{\text{ion}} - \dot{N}_{\text{rec}} \quad (1)$$

As in the standard Stromgren argument, we say that all ionizing photons emitted by the star are absorbed in the surrounding medium, thus  $\dot{N}_{\text{ion}}$  is simply the number of ionizing photons emitted per second, denoted by  $Q$

$$\dot{N}_{\text{ion}} = Q = \int_{\nu_t}^{\infty} \frac{L_{\nu}(\nu)}{h\nu} d\nu \quad (2)$$

Since we assume that all photons come out at a single frequency  $\nu_0$ , we simply have

$$Q = \frac{L}{h\nu_0} \quad (3)$$

The number of total radiative recombinations per second is

$$\dot{N}_{\text{rec}} = n_e n_p \alpha_B V \quad (4)$$

where  $n_e, n_p$  are the electron and proton number densities,  $V$  is the volume, and we will use the  $\alpha_B$  recombination coefficient (i.e., we will assume all ionizing photons are trapped in the HII region). Finally, the total number of atoms in the HII region is  $N_{\text{tot}} = nV$ . Assuming the HII region is essentially totally ionized,  $n_p = n_e = n$ , where the total number density  $n$  is assumed to be constant with radius. Then equation 1 can be written

$$n \frac{dV}{dt} = Q - n^2 \alpha_B V \quad (5)$$

which can be written

$$\frac{dV}{dt} = n\alpha_B \left[ \frac{Q}{n^2 \alpha_B} - V \right] \quad (6)$$

We see the characteristic scales are the recombination time  $t_{\text{rec}} = (n\alpha_B)^{-1}$  and the Stromgren volume,  $V_s = 4\pi R_s^3/3 = Q/n^2 \alpha_B$ , where the Stromgren radius is

$$R_s = \left[ \frac{3Q}{4\pi n^2 \alpha_B} \right]^{1/3} \quad (7)$$

So a nice way to write the differential equation is

$$\frac{dV}{dt} = -\frac{V_s}{t_{\text{rec}}} \left[ \frac{V}{V_s} - 1 \right] \quad (8)$$

The solution is

$$\ln(V/V_s - 1) = -t/t_{\text{rec}} + C \quad (9)$$

The integration constant  $C = 0$  since the volume is zero at  $t = 0$ , so

$$V(t) = V_s(1 - e^{-t/t_{\text{rec}}}) \quad (10)$$

or in terms of the radius

$$R(t) = R_s(1 - e^{-t/t_{\text{rec}}})^{1/3} \quad (11)$$

We see that as  $t \rightarrow \infty$ , the radius of the HII region goes to  $R_s$ , as expected. The timescale for the HII region to grow is given by the recombination timescale  $t_{\text{rec}} = 1/n\alpha_B$ .

**1b)** At a temperature  $T = 10^4$  K, the recombination coefficient is  $\alpha_B \approx 2 \times 10^{-13}$ . So plugging in numbers we find

$$t_{\text{rec}} = (n\alpha_B)^{-1} \approx 1.6 \times 10^5 \text{ years} \quad (12)$$

This is comparable to the lifetime of an O-star, so the HII region will just about grow to its Stromgren radius when the star is about to die.

**1c)** We can find the velocity of the edge of the HII region by differentiating our solution

$$v(t) = \frac{dR(t)}{dt} = \frac{R_s}{t_{\text{rec}}} \frac{e^{-t/t_{\text{rec}}}}{3} (1 - e^{-t/t_{\text{rec}}})^{-2/3} \quad (13)$$

We could have guessed that characteristic velocity of the HII region expansion is

$$v_s \sim R_s/t_{\text{rec}} \approx 10^7 \text{ cm s}^{-1} \quad (14)$$

This is about an order of magnitude greater than the sound speed  $c \sim (kT/m_p)^{1/2} \sim 10^6 \text{ cm s}^{-1}$ . Thus the HII region expansion is initially supersonic and we can neglect hydrodynamical effects.

**1d)** To determine the ionization state at a radius  $r$ , we apply the equation that expresses photoionization equilibrium

$$4\pi \int_0^\infty \frac{J_\nu(r)}{h\nu} \sigma(\nu) n_{\text{HI}} = n_e n_p \alpha_B \quad (15)$$

which states that the local photoionization rate equals the radiative recombination rate. Here  $\sigma(\nu)$  is the bound-free cross-section for hydrogen, and  $J_\nu(r)$  the monochromatic mean intensity of the radiation field at radius  $r$ . Assuming the radiation source is an isotropically emitting point source of intensity  $I_\nu$  and radius  $R_\star$ , and there is negligible attenuation above it (which should be OK at very small radii) we can use the standard result

$$J_\nu = \frac{I_\nu R_\star^2}{4r^2} \quad (16)$$

The monochromatic flux at the surface of a Lambert radiator is  $F_\nu = \pi I_\nu$  and so the monochromatic luminosity is

$$L_\nu = 4\pi R_\star^2 F_\nu = 4\pi^2 R_\star^2 I_\nu \quad (17)$$

So we can write the monochromatic mean intensity as

$$J_\nu = \frac{L_\nu}{16\pi^2 r^2} \quad (18)$$

Using this value, the photoionization equilibrium equation becomes

$$\frac{Q\sigma_0}{4\pi r^2} n_{\text{HI}} = n_e n_p \alpha_B \quad (19)$$

where  $\sigma_0 = \sigma(\nu_0)$ . Defining the ionized fraction  $x_{\text{H}} = n_{\text{HII}}/n$ , we have for pure hydrogen,  $n_e = n_p = x_{\text{H}}n$  and  $n_{\text{HI}} = (1 - x_{\text{H}})n$  and so

$$\frac{4\pi Qn}{r^2} (1 - x_{\text{H}}) = n^2 x_{\text{H}}^2 \alpha_B \quad (20)$$

Dividing both sides by  $n^2 \alpha_B$ , we can write this in dimensionless form

$$2\Omega(1 - x_{\text{H}}) = x_{\text{H}}^2 \quad \text{where } \Omega = \frac{2\pi Q\sigma_0}{nr^2 \alpha_B} \quad (21)$$

Where the dimensionless quantity  $\Omega$  is apparently the useful measure for how strong the ionization is. We can now solve the quadratic equation for  $x_H$

$$x_H = -\Omega + \sqrt{\Omega^2 + 2\Omega} \quad (22)$$

and hence

$$x_{HI} = 1 - x_H = 1 + \Omega - \sqrt{\Omega^2 + 2\Omega} \quad (23)$$

We should check our limits. For a very weak source ( $L \rightarrow 0$  at fixed  $r$ ), we have  $\Omega \rightarrow 0$  and find no ionization  $x_H = 0$ , as expected. For a very strong source ( $L \rightarrow \infty$  at fixed  $r$ ), we have  $\Omega \rightarrow \infty$ , and we need to take a little more care in taking the limit. We use a Taylor expansion in the small quantity  $2/\Omega$ :

$$x_H = -\Omega + \Omega(1 + 2/\Omega)^{1/2} \approx -\Omega + \Omega(1 + 1/\Omega + \dots) \approx 1 \quad (24)$$

As expected, the medium is totally ionized.

**1e)** To find the small  $r$  behavior, we note that this is the limit  $\Omega \rightarrow \infty$ , so we use an Taylor expansion similar to the above, but keep another term

$$x_{HI} = 1 + \Omega - \Omega(1 + 2/\Omega)^{1/2} \approx 1 + \Omega - \Omega(1 + 1/\Omega - 1/4\Omega^2 + \dots) \quad (25)$$

Thus to lowest order in  $1/\Omega$  we find

$$x_{HI} \approx \frac{1}{4\Omega} \propto \frac{r^2 n}{L} \quad (26)$$

## 2 Chilling in the Halo

Provided by Io Kleiser and Janos Botyanzski

### Problem 2

#### 2a

We have the cooling time for hydrogen from class:

$$t_c = 9 \times 10^{10} T^{1/2} n_1^{-1} \text{sec.} \quad (17)$$

The problem says the mass of hydrogen gas is on the order of the dark matter mass, so let's assume  $M$  is the mass of hydrogen. This gives us  $n_1 = \frac{M}{m_p} \frac{3}{4\pi R^3}$ . Setting  $t_c = t_{\text{dyn}}$  gives the expression

$$\left[ \frac{GM}{R^3} \right]^{-1/2} \approx 9 \times 10^{10} T_v^{1/2} \left[ \frac{4\pi m_p R^3}{3M} \right]. \quad (18)$$

Doing algebra and using the definition of the virial temperature to get  $M \sim k_B T_v R / G m_p$ , we find that the temperature cancels and we get the expression for  $R_g$ :

$$R_g \approx \frac{3k_B}{4\pi m_p^{3/2} 9 \times 10^{10}} = 2.16 \times 10^{23} \text{cm} = 70.2 \text{kpc}, \quad (19)$$

so we indeed find that the radius is on the order of 80 kpc.

#### 2b

Free-free emission is dependent on the amount of free electrons and protons present in the gas. Therefore, we need to calculate the fraction of ions given a certain temperature and the assumption of CIE (collisional ionization equilibrium):

$$n_{\text{HI}} C_{ic} = n_{\text{HII}} \alpha_A n_e \quad (20)$$

From this we can derive that

$$y_{\text{II}} = \frac{n_{\text{II}}}{n_{\text{I}}} = \frac{C_{ic}}{\alpha_A n_e} = 1.3 \times 10^{15} \left(\frac{k}{\chi}\right)^2 \left(\frac{T}{T_4}\right) e^{-\chi/kT} \quad (21)$$

And we can write

$$x_{\text{II}} = \frac{n_{\text{II}}}{n_{\text{H}}} = \frac{n_{\text{II}}}{n_{\text{I}} + n_{\text{II}}} = \frac{y_{\text{II}}}{1 + y_{\text{II}}} \quad (22)$$

These will be important when writing out the electron and proton densities in what follows.

The emission is given by

$$\epsilon_{\text{ff}} = 1.4 \times 10^{-27} Z^2 n_e n_{\text{II}} T^{1/2} g_{\text{ff}} \quad (23)$$

We have already calculated  $n_{\text{II}}$  above. Assume that  $g_{\text{ff}} \approx 1$ , set  $Z = 1$  for Hydrogen, and convert to dimensionless units. Then

$$\Lambda_{\text{ff}} = \frac{\epsilon}{n_e n_{\text{H}}} = 1.4 \times 10^{-25} \frac{y_{\text{II}}}{1 + y_{\text{II}}} \left(\frac{T}{T_4}\right)^{1/2} \frac{\text{erg cm}^3}{\text{s}} \quad (24)$$

For bound-free emission, again the proton and electron fractions are important. The emission is

$$\epsilon_{\text{bf}} = 3.25 \times 10^{-13} n_e n_{\text{I}} k_{\text{B}} T \left(\frac{T}{T_4}\right)^{-1/2} \quad (25)$$

Convert to dimensionless units. Then

$$\Lambda_{\text{bf}} = \frac{\epsilon}{n_e n_{\text{H}}} = 5 \times 10^{-25} \frac{y_{\text{II}}}{1 + y_{\text{II}}} \left(\frac{T}{T_4}\right)^{1/2} \frac{\text{erg cm}^3}{\text{s}} \quad (26)$$

For Lyman- $\alpha$  emission,

$$\epsilon_{\text{Ly-}\alpha} \approx h\nu_0 n_2 A_{21} = h\nu_0 n_1 C_{12} \quad (27)$$

where  $\nu_0$  is the frequency associated with the Lyman- $\alpha$  transition,  $A_{21}$  is the rate of transitions from  $n = 2 \rightarrow n = 1$ , and  $C_{12}$  is the rate of collisional excitation from  $n = 1 \rightarrow n = 2$ .

Using the formulae derived in class, we can rewrite  $C_{12}$  in the final expression to get

$$\epsilon_{\text{Ly}-\alpha} \approx 2.16 h \nu_0 n_1 n_e f \left( \frac{h \nu_0}{kT} \right)^{-1.68} T^{-3/2} e^{-h \nu_0 / kT} \quad (28)$$

where  $f \approx .5$  is the oscillator strength of the Lyman- $\alpha$  transition and  $n_1$  is the fraction of neutral Hydrogen atoms in the ground state (assume  $n_1 \approx n_{\text{I}}$ ). Converting into dimensionless variables, we have

$$\Lambda_{\text{Ly}-\alpha} = \frac{\epsilon}{n_e n_{\text{H}}} = 4 \times 10^{-18} \frac{y_{\text{II}}}{1 + y_{\text{II}}} \left( \frac{h \nu_0}{kT} \right)^{-.68} \left( \frac{T}{T_4} \right)^{-1/2} \frac{\text{erg cm}^3}{\text{s}} \quad (29)$$

To calculate the fraction of neutral Hydrogen, write

$$\frac{n_{\text{I}}}{n_{\text{tot}}} = \frac{n_{\text{I}}}{n_{\text{I}} + n_{\text{II}}} = \frac{1}{1 + y_{n_{\text{II}}}} \quad (30)$$

So putting these together,

$$\boxed{\Lambda_{\text{Ly}-\alpha} = \frac{\epsilon}{n_e n_{\text{H}}} = 4 \times 10^{-18} \frac{1}{1 + y_{n_{\text{II}}}} \left( \frac{h \nu_0}{kT} \right)^{-.68} \left( \frac{T}{T_4} \right)^{-1/2} \frac{\text{erg cm}^3}{\text{s}}} \quad (31)$$

## 2c

The probability of absorption into the thermal pool is just the probability of collisional de-excitation. We can write this as

$$P = \frac{C_{21}}{C_{21} + A_{21}} \quad (32)$$

where  $A_{21} = 6.3 \times 10^8 \text{s}^{-1}$  (from NIST) and at  $T = T_4$ ,  $n_e = 1 \text{cm}^{-3}$ ,

$$C_{21} = 5 \times 10^{-3} \text{s}^{-1} \quad (33)$$

Plugging these values into the above equation, we have

$$P = \frac{C_{21}}{C_{21} + A_{21}} \approx \boxed{10^{-11}} \quad (34)$$

So there is a very small probability that absorption into the thermal pool will take place.

## 2d

We solve the following balance equations

$$n_5 C_{56} = n_6 R_{65} \quad (35)$$

$$n_6 C_{67} = n_7 R_{76} \quad (36)$$

$$n_{\text{oxy}} = n_5 + n_6 + n_7 \quad (37)$$

First divide the latter equation by  $n_6$  and invert to get

$$\frac{n_6}{n_{\text{oxy}}} = \left[ \frac{n_5}{n_6} + 1 + \frac{n_7}{n_6} \right]^{-1} \quad (38)$$

Now plug in the relations from above to arrive at  $\boxed{\frac{n_6}{n_{\text{oxy}}} = \left[ \frac{R_{65}}{C_{56}} + 1 + \frac{C_{67}}{R_{76}} \right]^{-1}}$

## 2e

Look at the Grotrian diagram for the OVI ion. You can see that the first excited state is a short energy step above ground state ( $\Delta E = 12\text{eV}$ ). From this we can confidently say that mostly the ground and first excited states will be populated. The cooling will therefore be dominated by the transitions  $n = 2 \rightarrow n = 1$ .

For OVII, the lowest energy level transition requires a  $\lambda = 22A$  photon to excite the electron to that level,, or a gas temperature of  $\Delta E/k = 6 \times 10^6$  Kelvins. This is outside of our temperature range, so we can neglect this temperature.

For OVIII, the lowest energy level transition requires a  $\lambda = 18A$  photon to excite the electron to that level,, or a gas temperature of  $\Delta E/k = 8 \times 10^6$  Kelvins. This is outside of our temperature range, so we can neglect this temperature.

But a transition for OV only requires a  $\lambda = 1218A$  photon to excite the electron to that level, or a gas temperature of  $\Delta E/k = 1 \times 10^5$  Kelvins. Plenty of particles will have enough energy to impart collisionally to excite this transition.

## 2f

Here we derive the cooling function for the line transition of OVII. We assume CIE again. Start with the expression for emission:

$$\epsilon_{\text{OVI}} = n_2 A_{21} h\nu_0 \quad (39)$$

where  $\nu_0$  is the frequency associated with the line transition and  $n_2$  is the number density of particles in the excited state of OVI. Assuming collisional de-excitation is a negligible process. Then

$$n_2 A_{21} = n_1 C_{12} = (n_{\text{OVI}} - n_2) C_{12} \quad (40)$$

Rearranging:

$$n_2 = n_{\text{OVI}} \frac{C_{12}}{C_{12} + A_{21}} \quad (41)$$

And recall from section (2d)

$$\frac{n_{\text{OVI}}}{n_{\text{oxy}}} = \left[ \frac{R_{65}}{C_{56}} + 1 + \frac{C_{67}}{R_{76}} \right]^{-1} \quad (42)$$

Assuming solar abundances, we can make the following conversion

$$\frac{n_{\text{OVI}}}{n_{\text{oxy}}} = \frac{X_{\text{H,sol}}}{X_{\text{oxy,sol}}} \frac{n_{\text{OVI}}}{n_{\text{H}}} \quad (43)$$

where we have defined  $X_{\text{oxy,sol}} = n_{\text{oxy}}/n_{\text{tot}}$  in the sun. Thus, finally, we can write

$$n_2 = n_{\text{H}} \frac{X_{\text{oxy,sol}}}{X_{\text{H,sol}}} \left[ \frac{R_{65}}{C_{56}} + 1 + \frac{C_{67}}{R_{76}} \right]^{-1} \frac{C_{12}}{C_{12} + A_{21}} \quad (44)$$

Putting it all together,

$$\Lambda_{\text{OVI}} = \frac{\epsilon}{n_e n_{\text{H}}} = h\nu_0 A_{21} \frac{X_{\text{oxy,sol}}}{X_{\text{H,sol}}} \left[ \frac{R_{65}}{C_{56}} + 1 + \frac{C_{67}}{R_{76}} \right]^{-1} \frac{C_{12}}{C_{12} + A_{21}} \frac{\text{erg cm}^3}{\text{s}} \quad (45)$$

where

$$C_{12} \approx 3.9 n_e f \left[ \frac{h\nu_0}{kT} \right]^{-1} T^{-3/2} e^{-h\nu_0/kT}$$

for  $\nu_0$  corresponding to the line transition,

$$C_{56} = 2.7n_e \left[ \frac{\chi_{56}}{kT} \right] T^{-3/2} e^{-\chi_{56}/kT}$$

$$C_{67} = 2.7n_e \left[ \frac{\chi_{67}}{kT} \right] T^{-3/2} e^{-\chi_{67}/kT}$$

$$R_{65} = 2 \times 10^{-13} Z^2 (T/T_4)^{-1/2}$$

for  $Z=7$ ,

$$R_{76} = 2 \times 10^{-13} Z^2 (T/T_4)^{-1/2}$$

for  $Z=6$ .

The cooling functions from section (2b) and (2f) are plotted in Fig. 1.

Figure 1: Cooling function. Courtesy of J.L. Barnes.

