Random interactions, random matrices, and emergent symmetries

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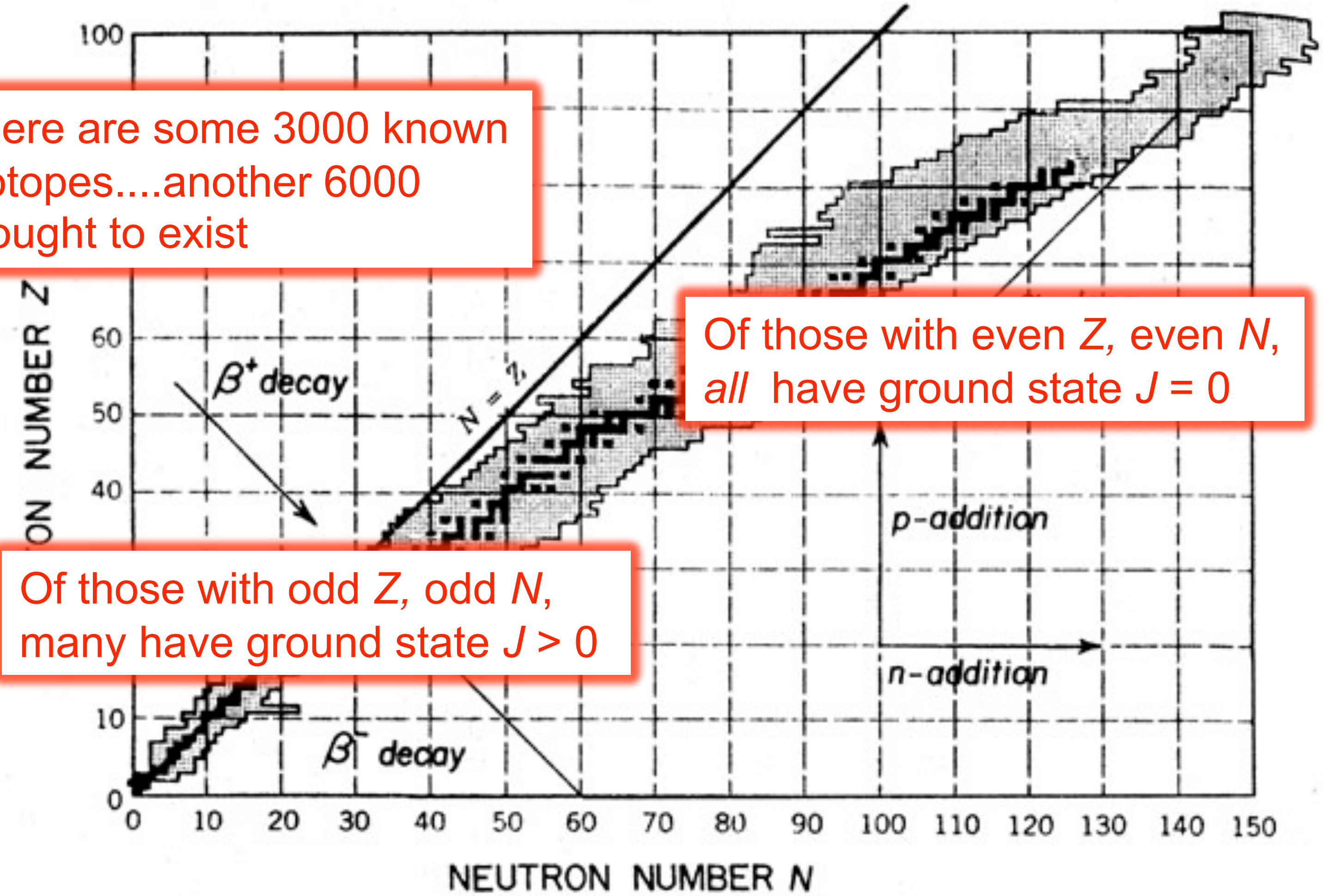
C. W. J, G. F. Bertsch, and D.J.Dean , Phys. Rev. Lett. **80** (1998) 2749.

C. W. J, G. F. Bertsch, D. J. Dean, and I. Talmi, Phys. Rev. C **61**, 014311 (2000).

BertschFest 2012

The origin of nuclear spin

There are some 3000 known isotopes....another 6000 thought to exist



Of those with even *Z*, even *N*, *all* have ground state *J* = 0

Of those with odd *Z*, odd *N*, many have ground state *J* > 0

Why?

The origin of nuclear spin

What is behind these regularities?

Textbook answer: the nuclear force, e.g.,
"pairing" force + quadrupole-quadrupole

We will test this *hypothesis* by running simulations
with an ensemble of "random" interactions

The origin of nuclear spin

Nuclear Hamiltonian: $$\hat{H} = \sum_i -\frac{\hbar^2}{2M}\nabla_i^2 + \sum_{i<j} V(r_i, r_j)$$

Solve by diagonalizing **H** in a basis of many-body states.
The many-body states are *Slater determinants*, or
anti-symmeterized products of single-particle wfns.

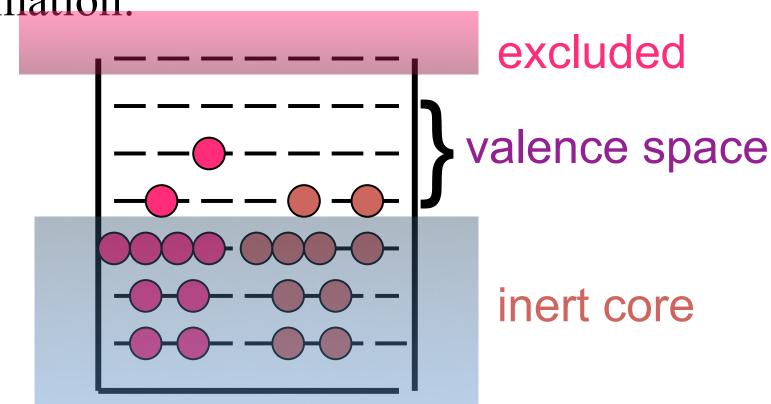
At this point one generally goes to occupation representation:

$$\hat{H} = \sum_i \varepsilon_i \hat{a}_i^+ \hat{a}_i + \frac{1}{4}\sum_{ijkl} V_{ijkl} \hat{a}_i^+ \hat{a}_j^+ \hat{a}_l \hat{a}_k$$

single-particle energies

two-body matrix elements

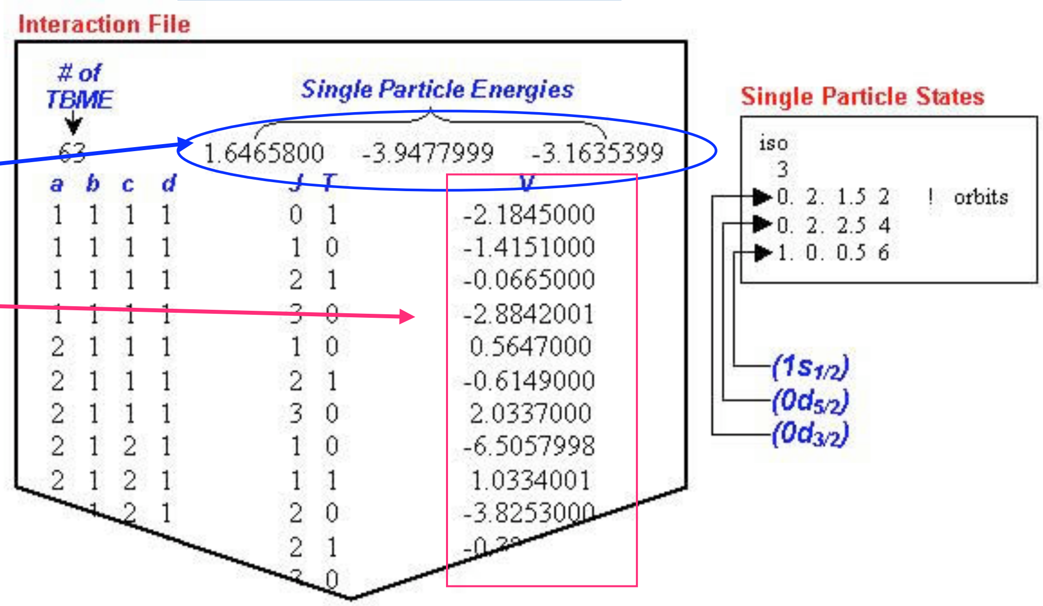
When running a fermion shell model code (e.g. MFD, NuShell(X) BIGSTICK), one enters the following information:



(1) The single-particle valence space (such as *sd* or *pf*); assumes inert core

(2) The many-body model space (number of protons and neutrons, truncations, etc.)

(3) The interaction:
single-particle energies
and
two-body matrix elements
$V_{JT}(ab,cd)$



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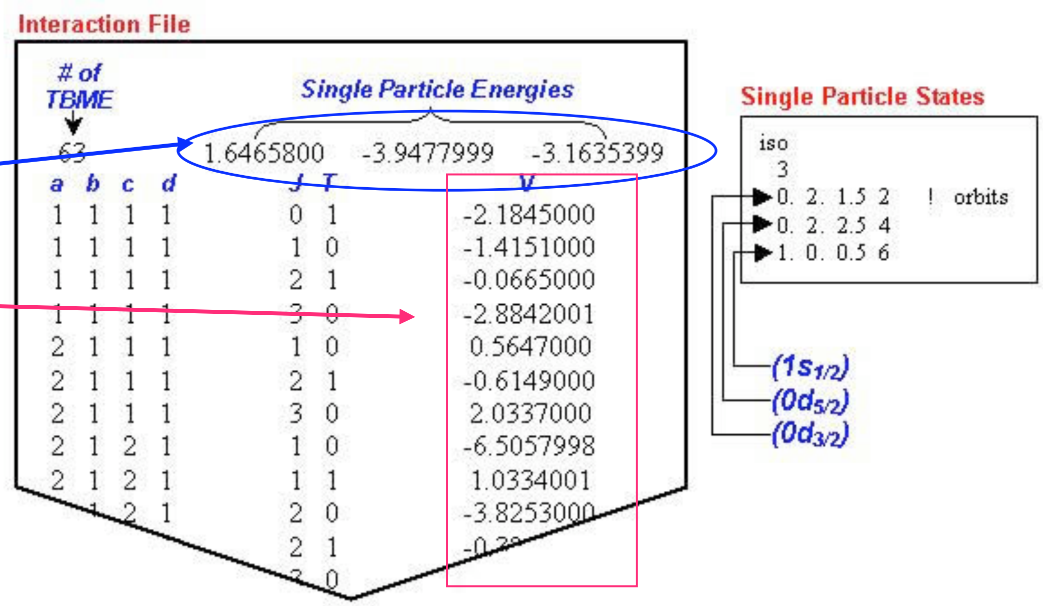
The two-body matrix elements *in principle but not in practice* depend on the single-particle wfns:

$$\left\langle ab;JT\middle|\hat{H}\middle|cd;JT\right\rangle = \int d^3r \int d^3r' \varphi_a{}^*(r)\varphi_b{}^*(r')\, V(r,r')\big(\varphi_c(r)\varphi_d(r') - \varphi_c(r')\varphi_d(r)\big)$$

But only the final number is read in!

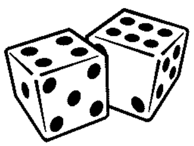
(3) The interaction:
single-particle energies
and
two-body matrix elements
$V_{JT}(ab,cd)$



Hence we can could randomly replace the carefully calculated two-body matrix elements with random numbers

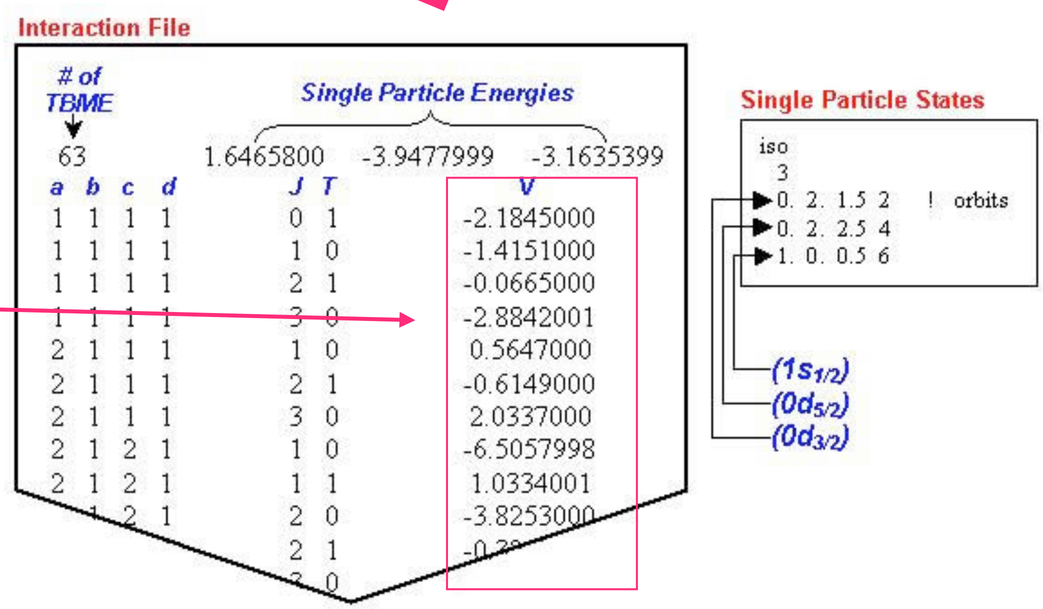
$$\left\langle ab;JT\left|\hat{H}\right|cd;JT\right\rangle = \int d^3r \int d^3r' \varphi_a{}^*(r)\varphi_b{}^*(r')$$

$$V(r,r')\left(\varphi_c(r)\varphi_d(r') - \varphi_c(r')\varphi_d(r)\right)$$

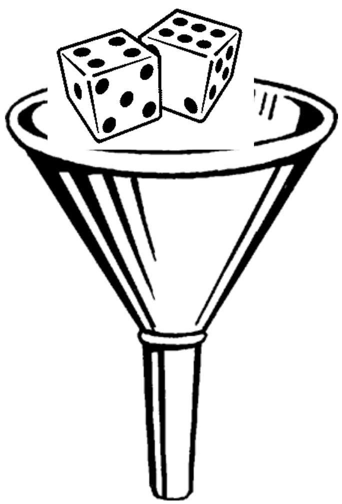


randomly drawn from
a Gaussian distribution*
(symmetric about zero)

*not very sensitive to details of distribution



A numerical experiment:



Draw interaction from
"two-body random ensemble"

General many-body
structure code

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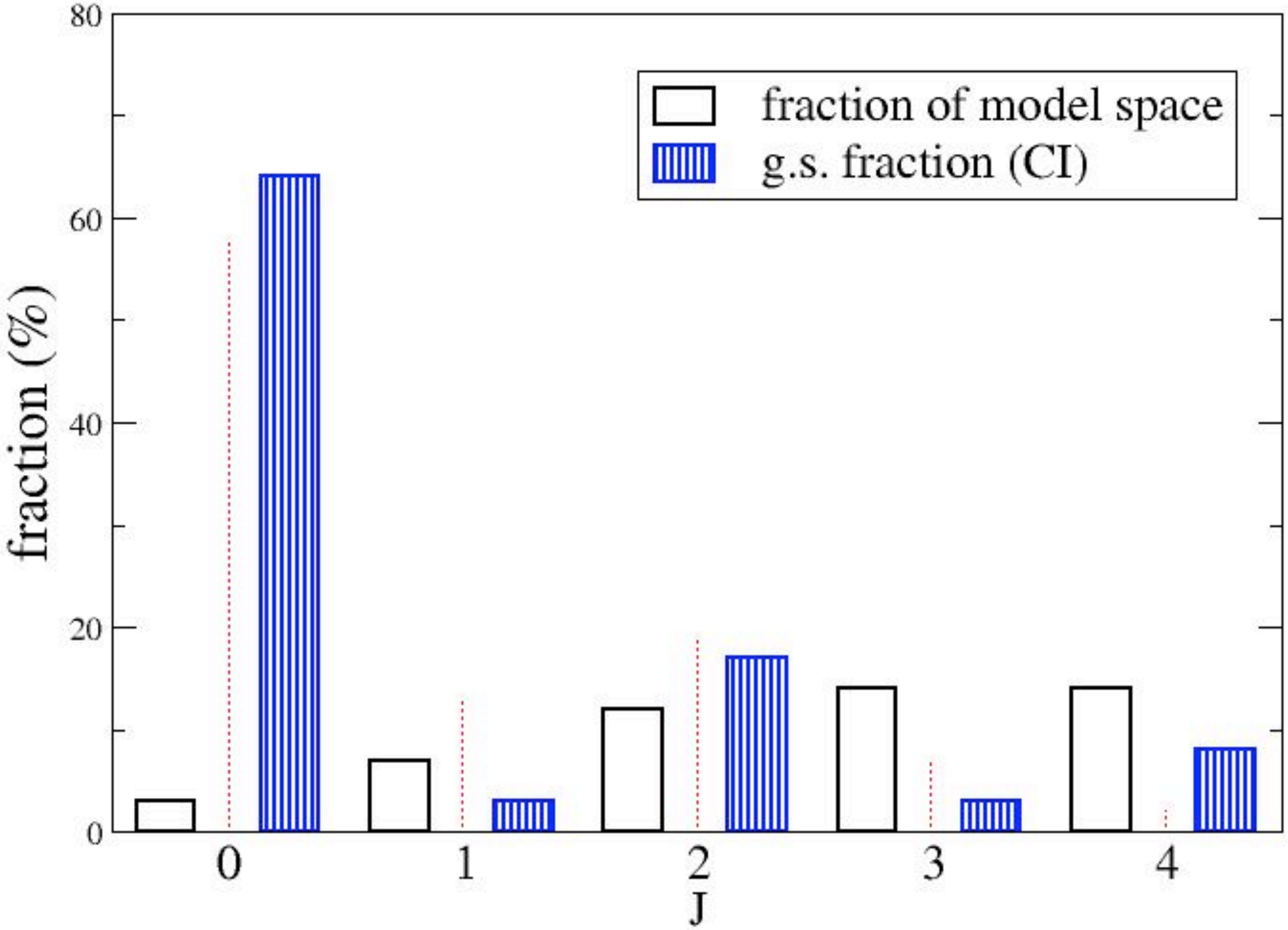

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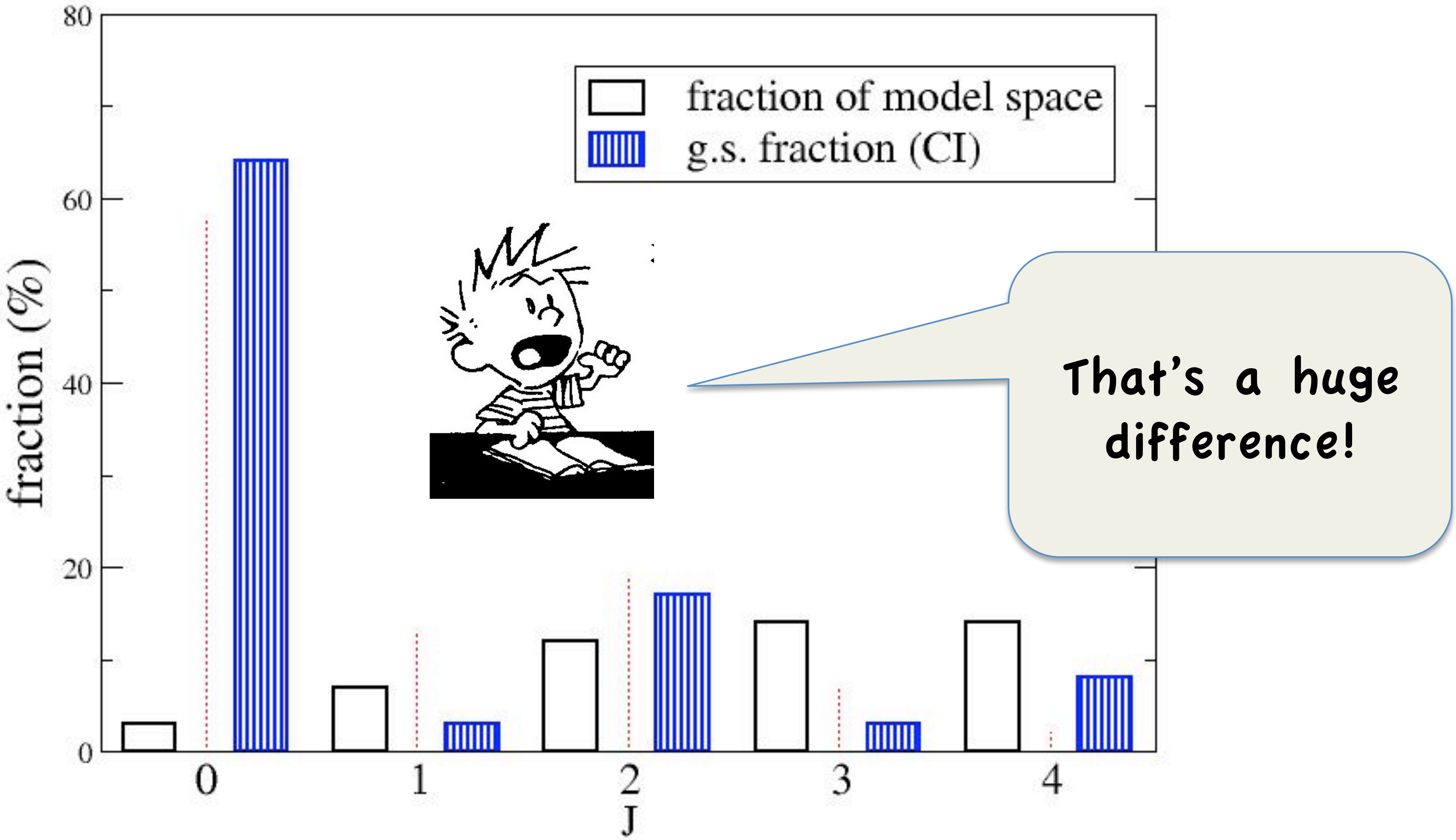

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We have a long list of results that *qualitatively* resemble nuclear structure:

- Pairing-like "gap" from g.s.
- Odd-even staggering
- One-particle, one-hole collectivity among low-lying states (band structure)

- Mallman plots for J = 0,2,4,6,8 states

The origin of nuclear spin

This is *amazing*!

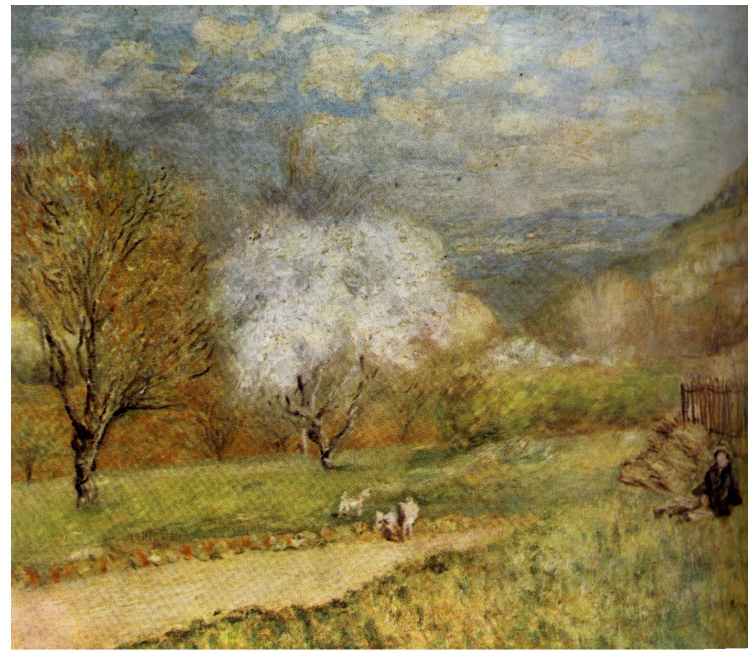
Do we understand this?

"...the simple question of symmetry and chaos asks for a simple answer which is still missing."

\- A. Volya, PRL **100**, 162501 (2008).

The origin of nuclear spin

Bellini, *Madonna and Child*



Renoir, *Country Road*

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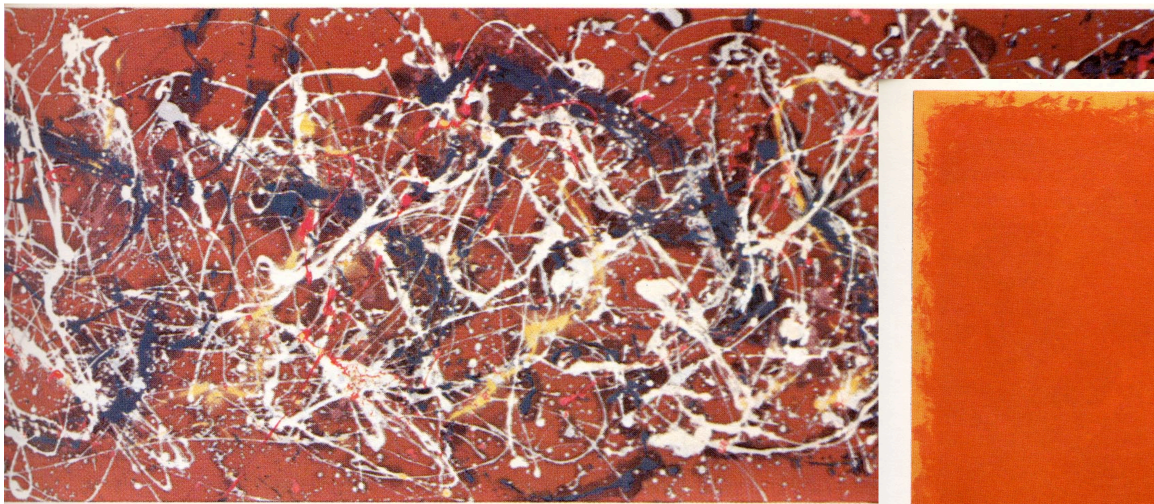
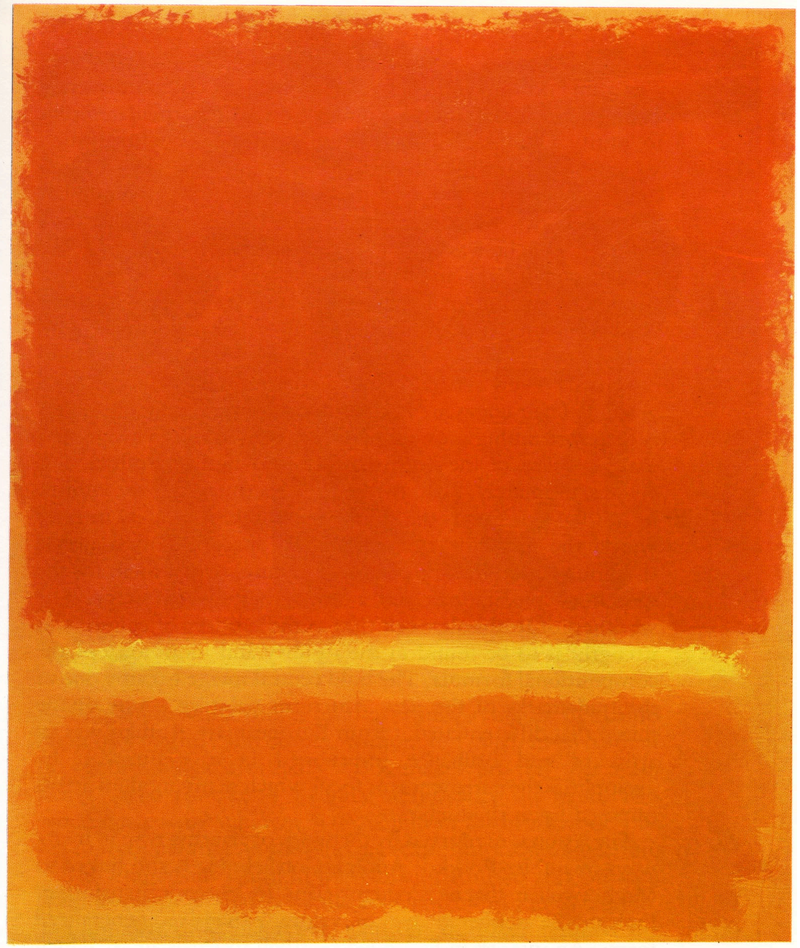


14 JACKSON POLLOCK *Number 2* 1949



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How can we think about many-body systems more simply? Especially in terms of quantum numbers like J and π?

We can look at *statistical averages* of Hamiltonian submatrices with good J and π,
and also look at correlations between them.

(This follows the ideas of J. Bruce French etc.;
also some of my results are similar to those developed by H. Weidenmüller and T. Papenbrock)

Start with some Hamiltonian matrix in a basis (e.g. a shell-model basis)

$$H = \begin{pmatrix} A & B & C & D & E & \cdots \\ B & F & G & H & I & \cdots \\ C & G & J & K & I & \\ D & H & K & L & M & \\ E & I & L & M & N & \\ \vdots & & & & & \ddots \end{pmatrix}$$

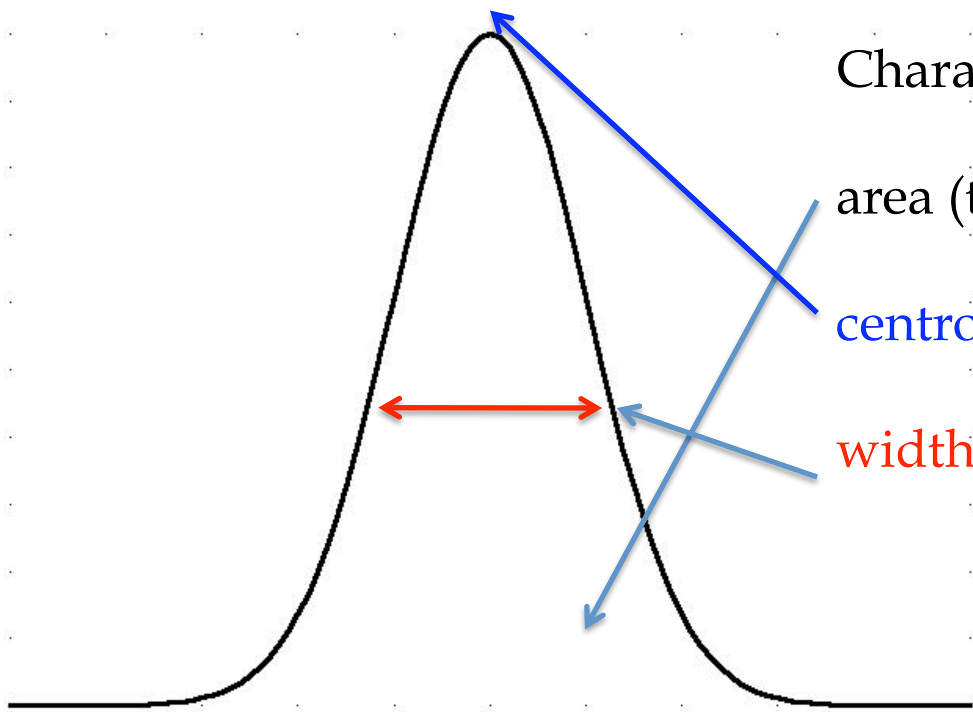
If **H** commutes with some symmetry generators, then we can bring it into block diagonal form...

Start with some Hamiltonian matrix in a basis (e.g. a shell-model basis)

$$H' = \begin{pmatrix} h_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & h_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & h_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & h_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & h_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & h_5 \end{pmatrix}$$

If **H** commutes with some symmetry generators, then we can bring it into block diagonal form...

If we calculate the distribution of eigenstates, the density of states tends to be Gaussian…



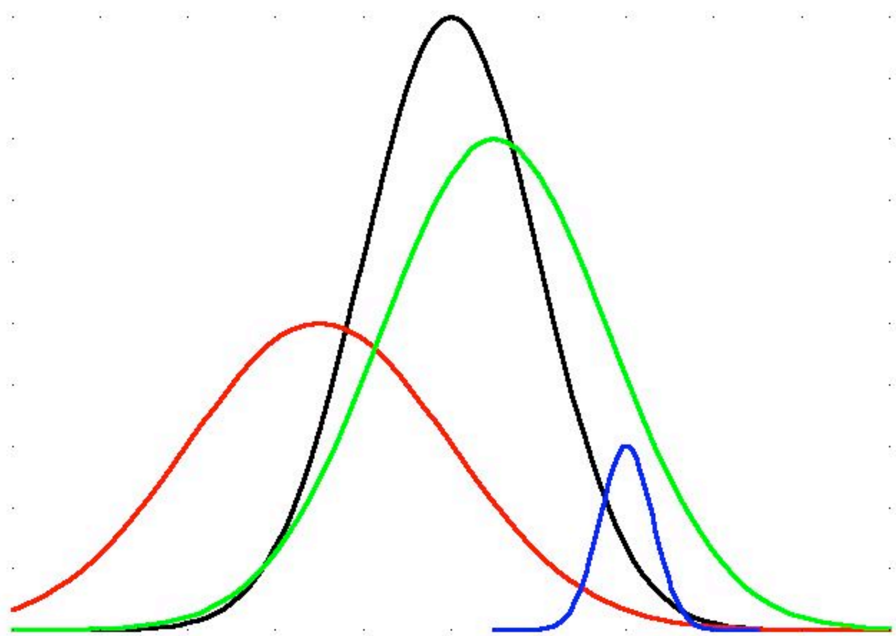
Characterized by:

area (total # of states)

centroid (average)

width

The distribution for each J also *tends* to be Gaussian, so we might use this to guess the g.s. quantum numbers



So we need the width, average, and dimension for different J...

I'm interested in general behavior of
many-body systems, so let's look at diverse cases:

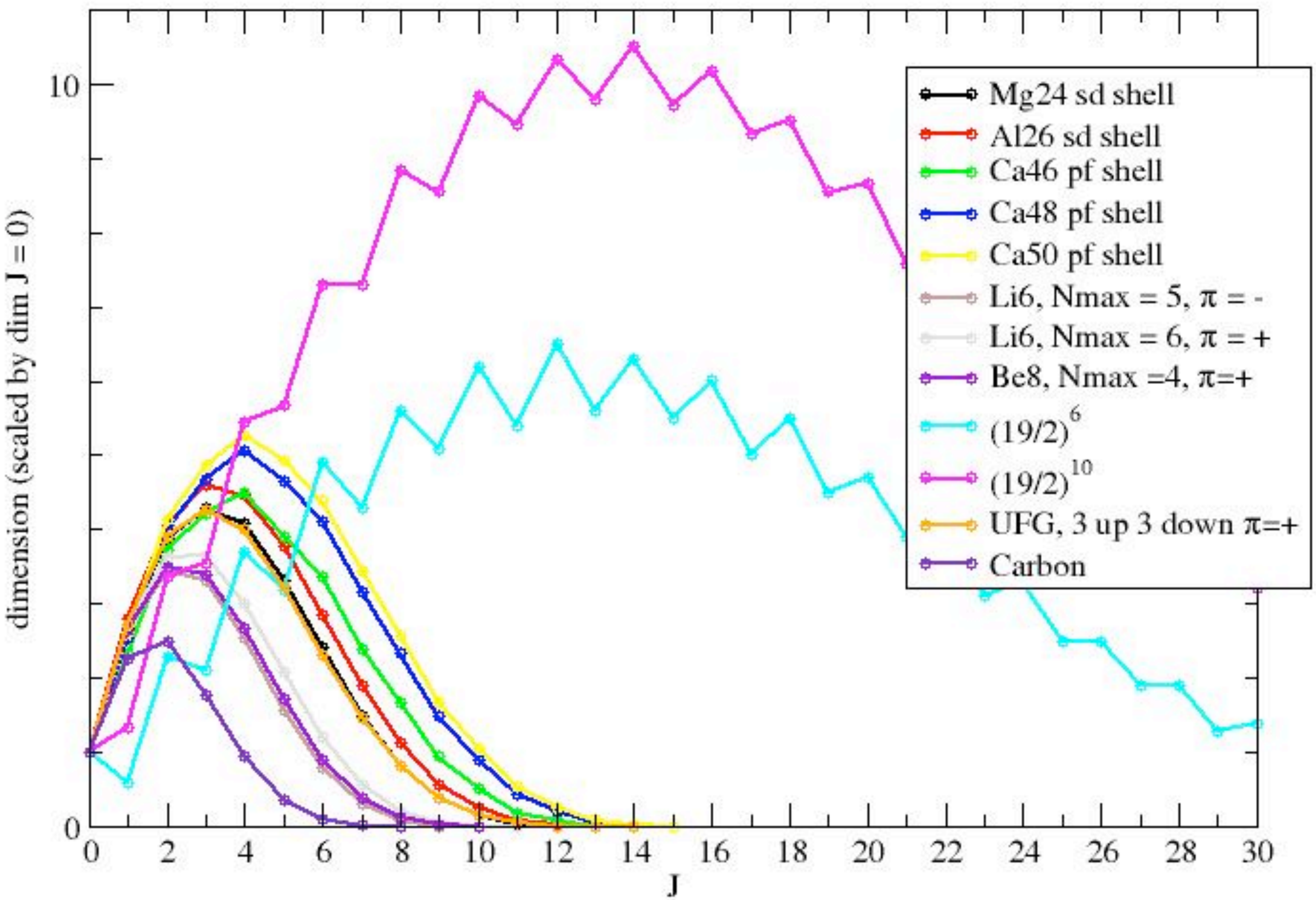
* *sd* and *pf* shell spaces with
  - -- "realistic" interactions (USDB and KB3G)
  - -- pure quadrupole-quadrupole
  - -- pure pairing
  - -- random two-body interactions
* no-core shell model spaces (6hw 6,7Li and 4hw 8Be) with *ab initio* interaction (NNLO evolved by SRG)
* single-*j* shell shell with random interactions, pure pairing
* "spin-chain" $(s_{1/2})^n$ with random interactions, pure pairing
* Unitary fermi gas (renormalized via Alhassid-Bertch-Fang)
* electronic structure of atoms: 4 to 6 electrons around a Carbon nucleus

J-projected dimensions:

$$N_J = trP_J$$

Scaled dimensions:

$$N_J / N_0$$



dimension (scaled by dim J = 0)
10
0
0
2
4
6
8
10
12
14
16
18
20
22
24
26
28
30
J
Mg24 sd shell
Al26 sd shell
Ca46 pf shell
Ca48 pf shell
Ca50 pf shell
Li6, Nmax = 5, π = -
Li6, Nmax = 6, π = +
Be8, Nmax =4, π=+
(19/2)^6
(19/2)^10
UFG, 3 up 3 down π=+
Carbon

J-projected centroids:

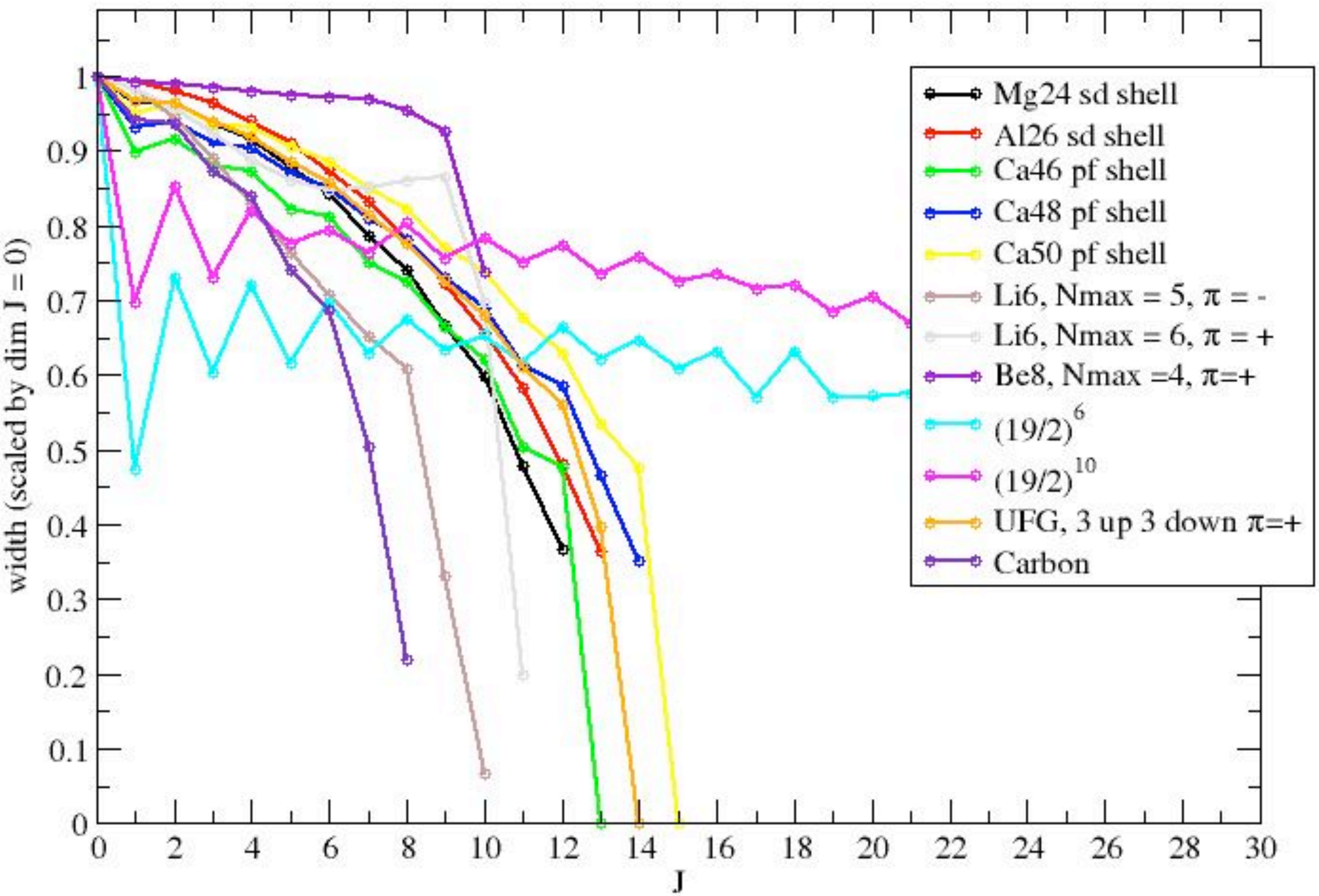
$$d_J = \frac{1}{N_J} tr\left(P_J H\right)$$

J-projected widths:

$$\sigma_J^2 = \frac{1}{N_J} tr\left(P_J H^2\right) - d_J^2$$

scaled widths:

$$\sigma_J / \sigma_0$$

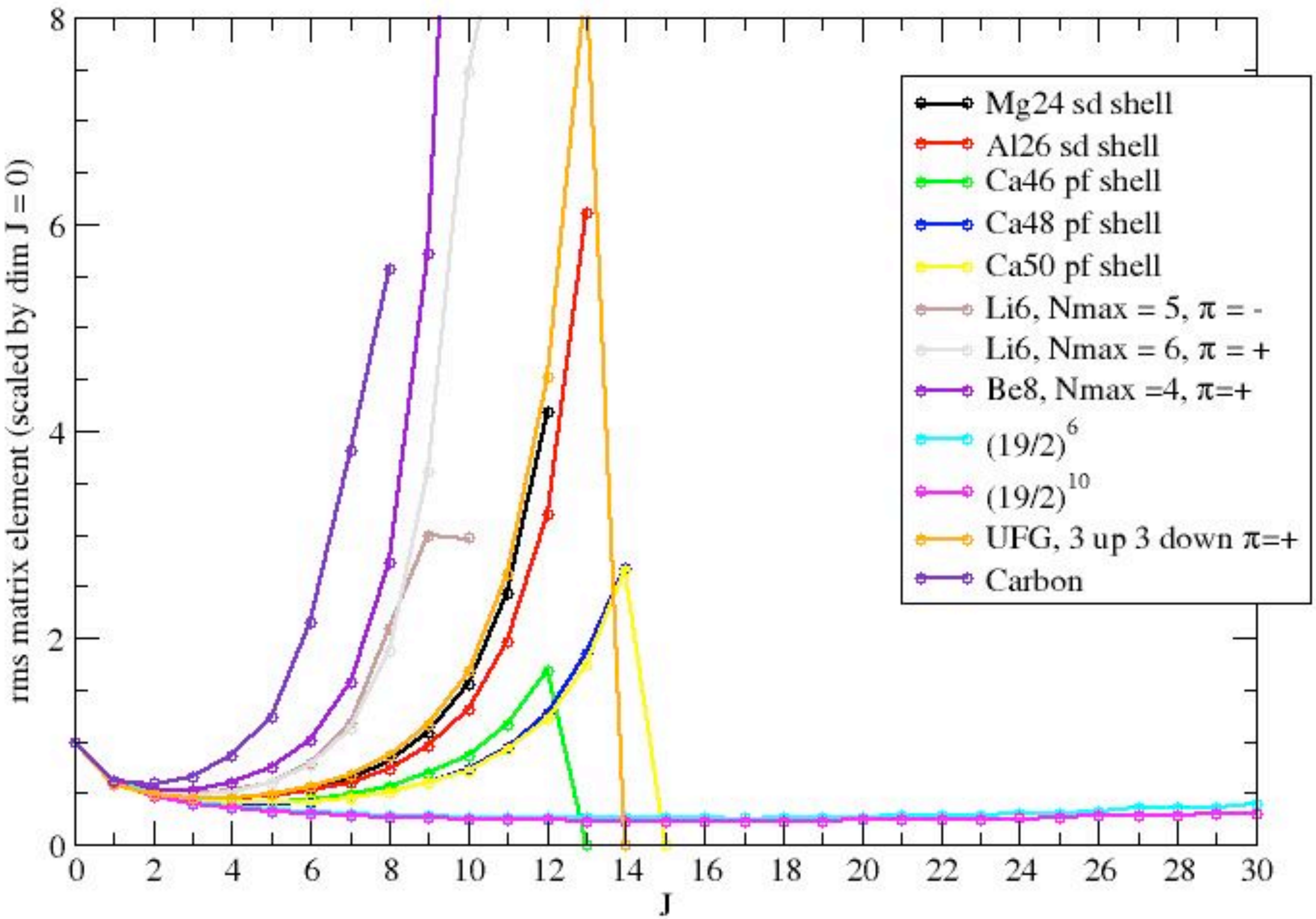

width (scaled by dim J = 0)
J
Mg24 sd shell
Al26 sd shell
Ca46 pf shell
Ca48 pf shell
Ca50 pf shell
Li6, Nmax = 5, π = -
Li6, Nmax = 6, π = +
Be8, Nmax =4, π=+
(19/2)^6
(19/2)^10
UFG, 3 up 3 down π=+
Carbon

rms matrix elements (of Hamiltonian)

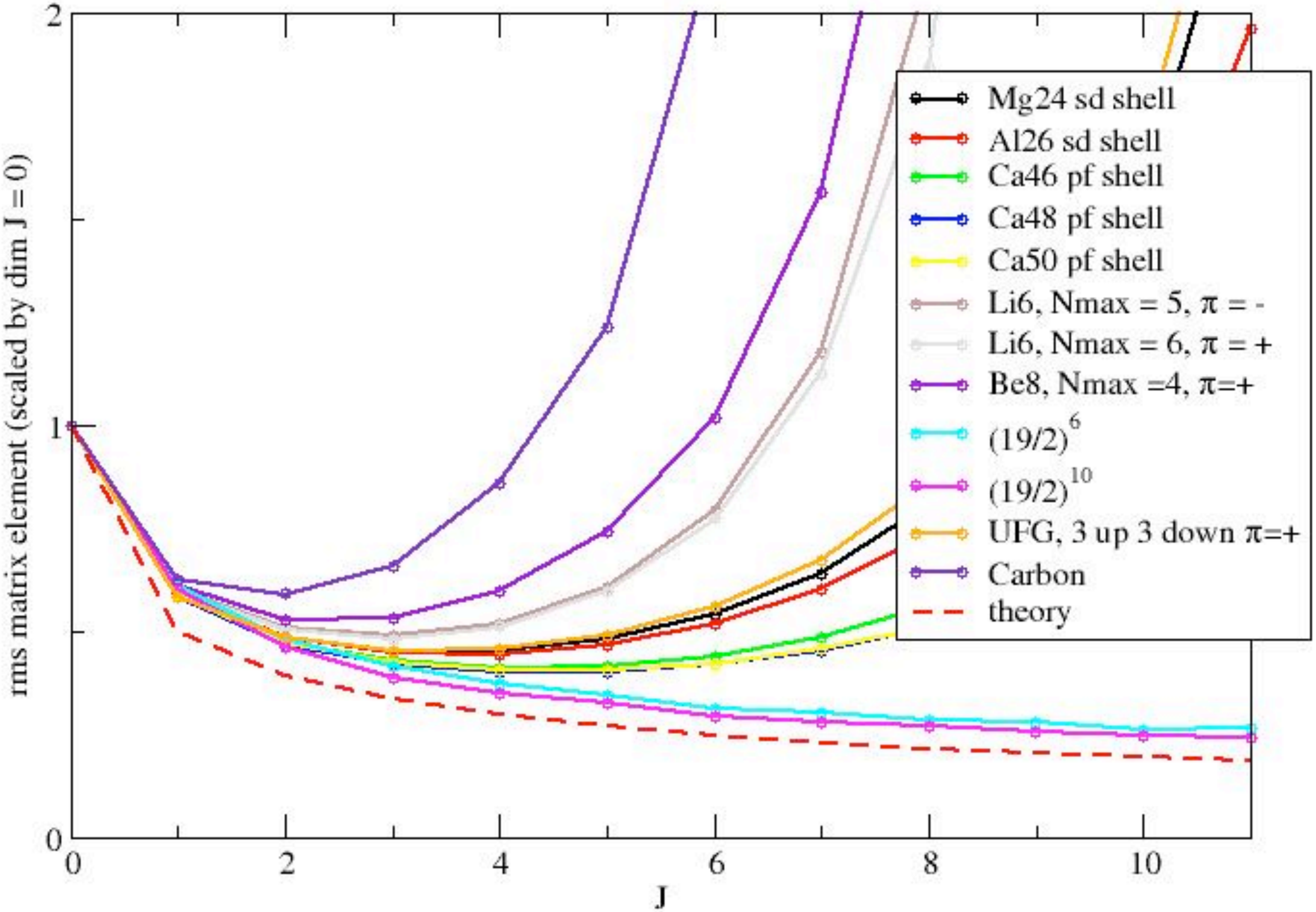
$$\gamma_J^2 = \frac{\sigma_j^2}{N_J} \approx \frac{1}{N_J^2} tr\left(P_J H^2\right)$$

Scaled rms matrix element:

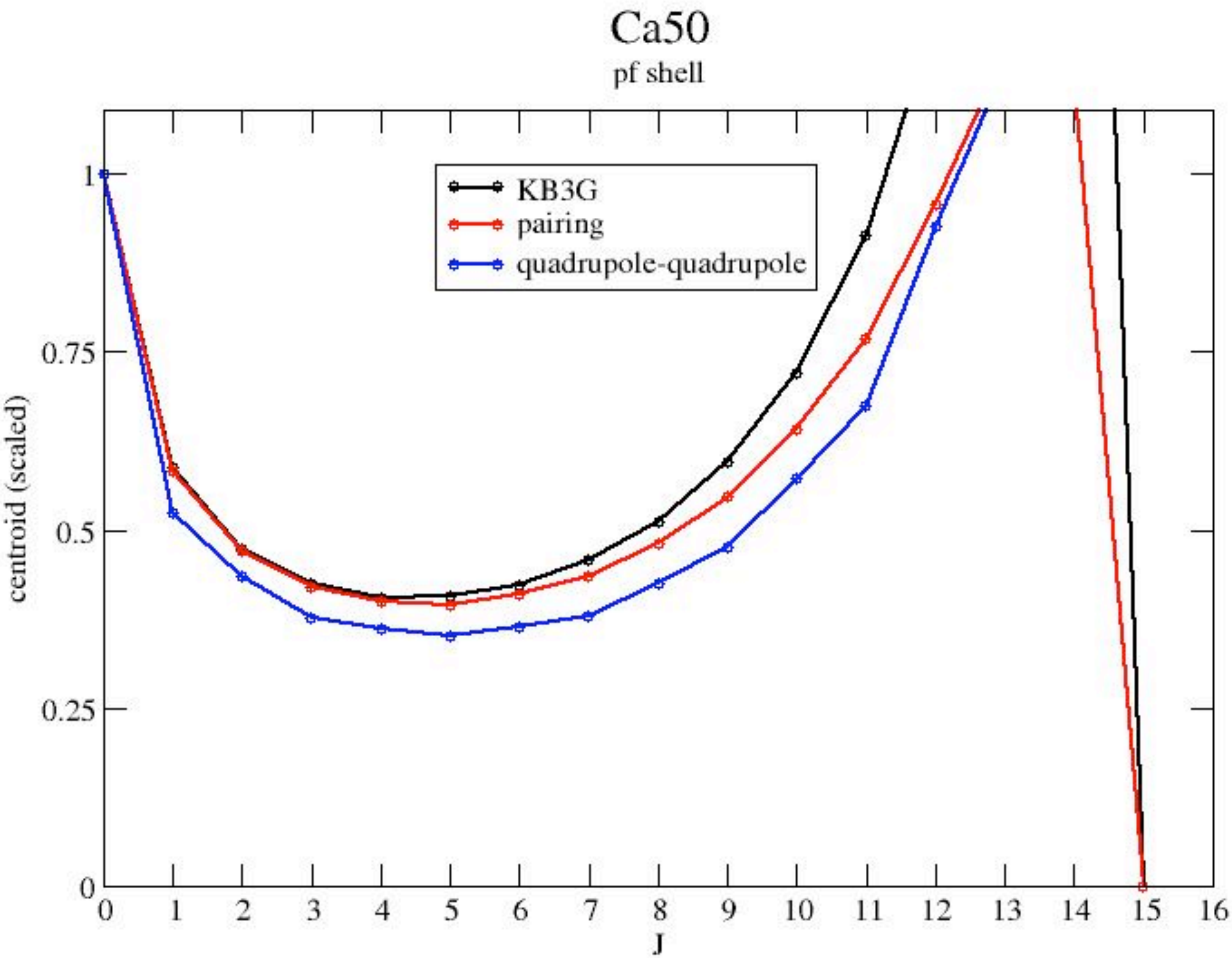
$$\gamma_J / \gamma_0$$



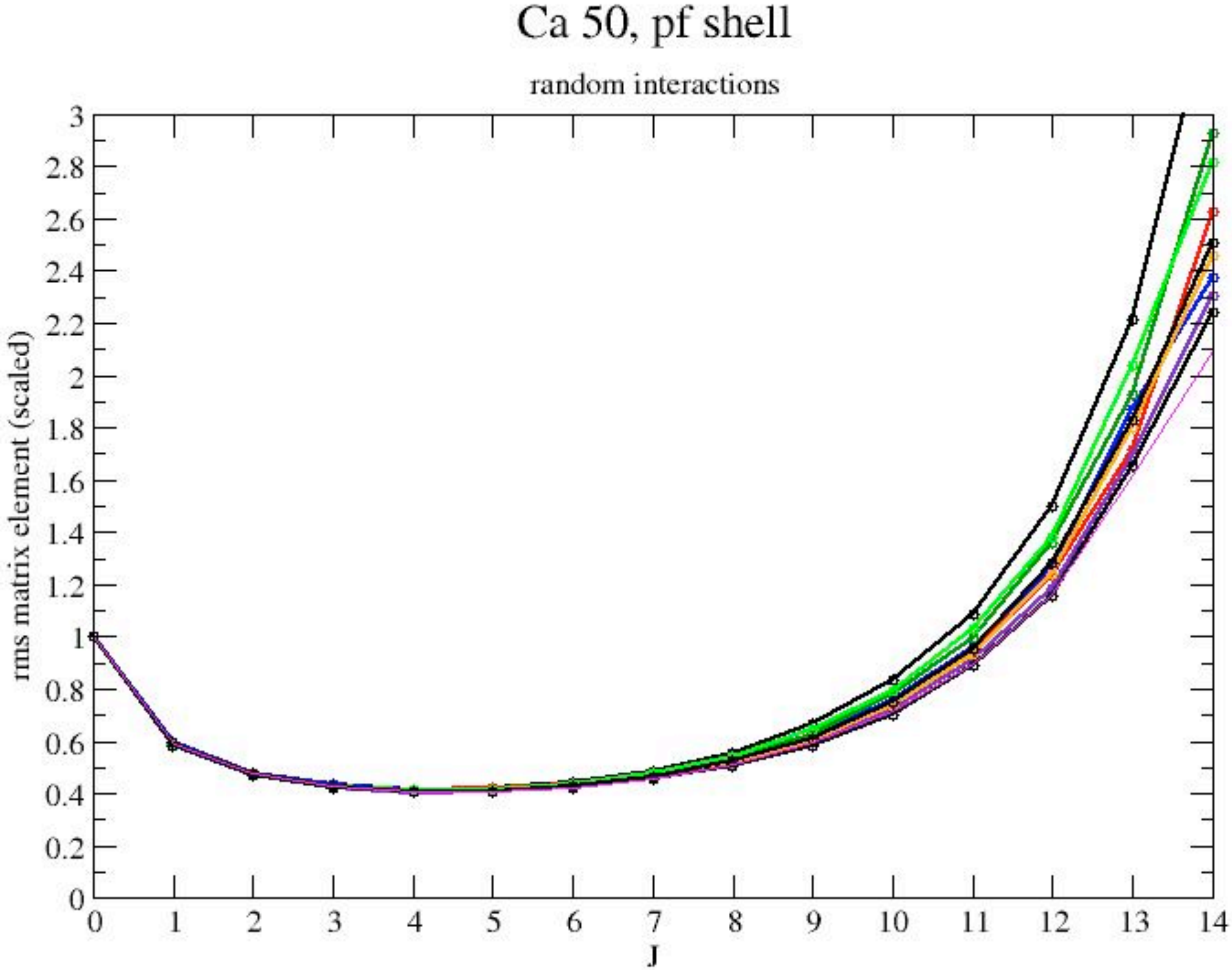
rms matrix element (scaled by dim J = 0)
8
6
4
2
0
0
2
4
6
8
10
12
14
16
18
20
22
24
26
28
30
J
Mg24 sd shell
Al26 sd shell
Ca46 pf shell
Ca48 pf shell
Ca50 pf shell
Li6, Nmax = 5, π = -
Li6, Nmax = 6, π = +
Be8, Nmax =4, π=+
(19/2)^6
(19/2)^10
UFG, 3 up 3 down π=+
Carbon



2
1
0
rms matrix element (scaled by dim J = 0)
0
2
4
6
8
10
J
Mg24 sd shell
Al26 sd shell
Ca46 pf shell
Ca48 pf shell
Ca50 pf shell
Li6, Nmax = 5, π = -
Li6, Nmax = 6, π = +
Be8, Nmax =4, π=+
(19/2)^6
(19/2)^10
UFG, 3 up 3 down π=+
Carbon
theory



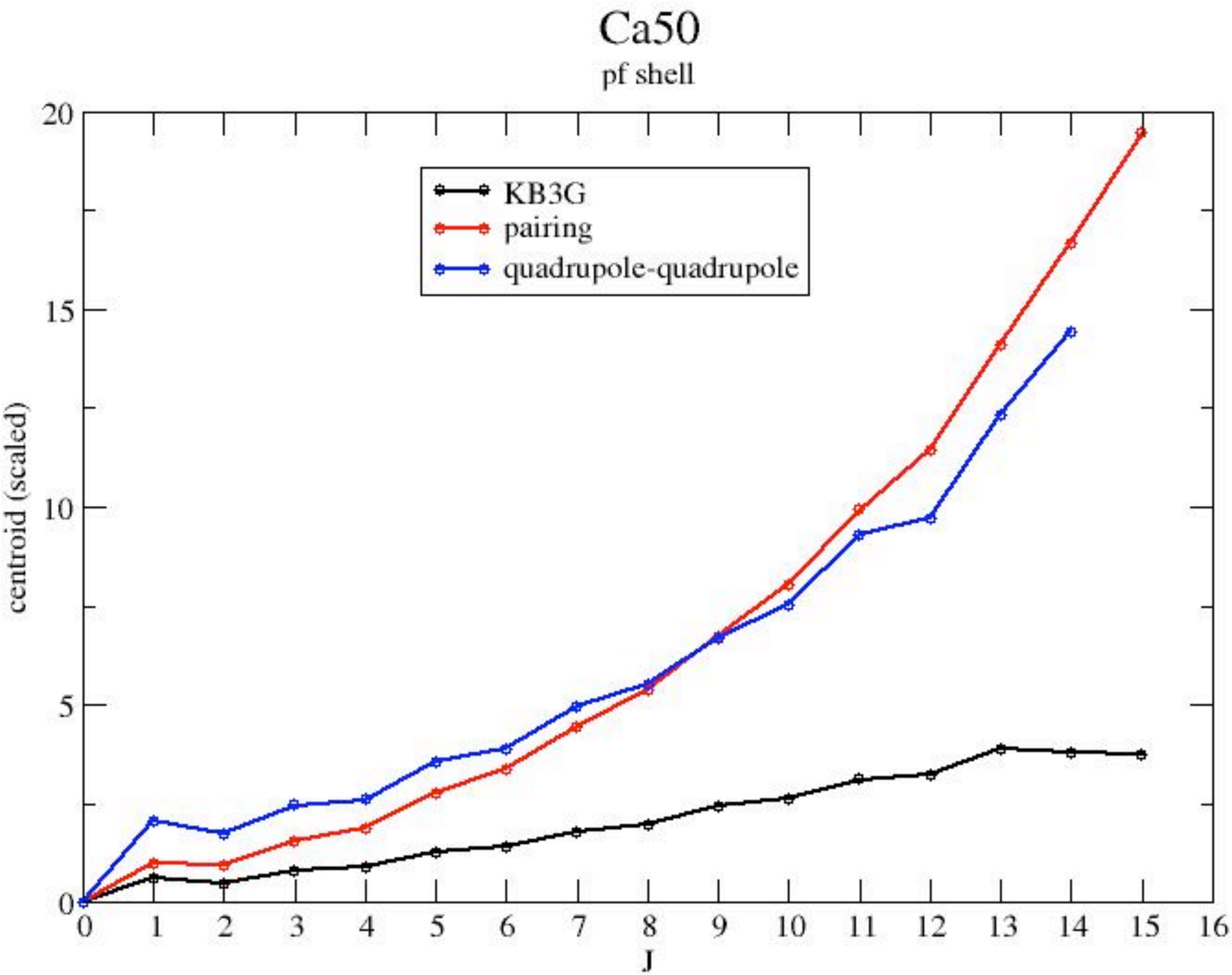
Ca50
pf shell
KB3G
pairing
quadrupole-quadrupole
centroid (scaled)
0
0.25
0.5
0.75
1
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
J

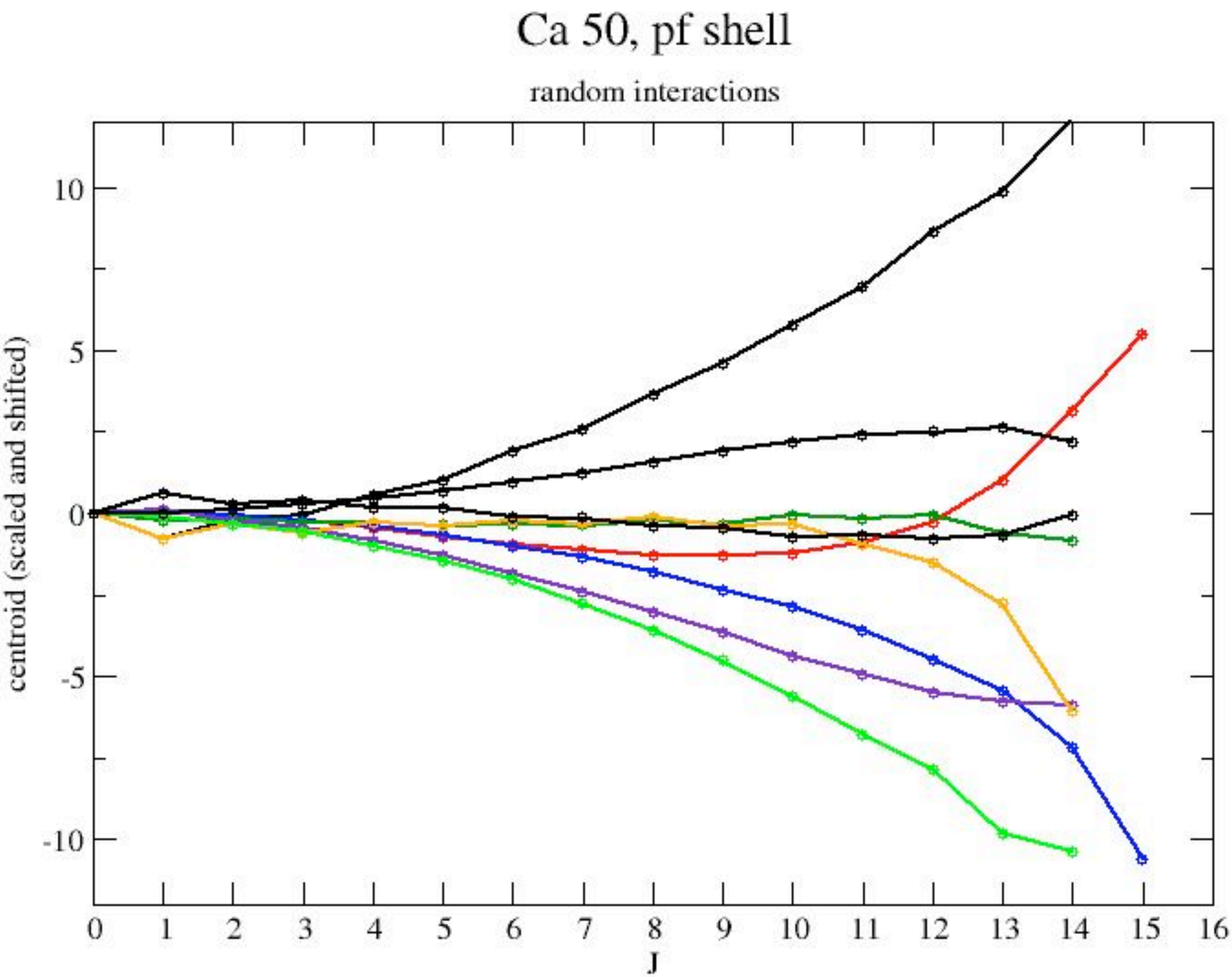


Ca 50, pf shell
random interactions
rms matrix element (scaled)
J
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
2.2
2.4
2.6
2.8
3
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14

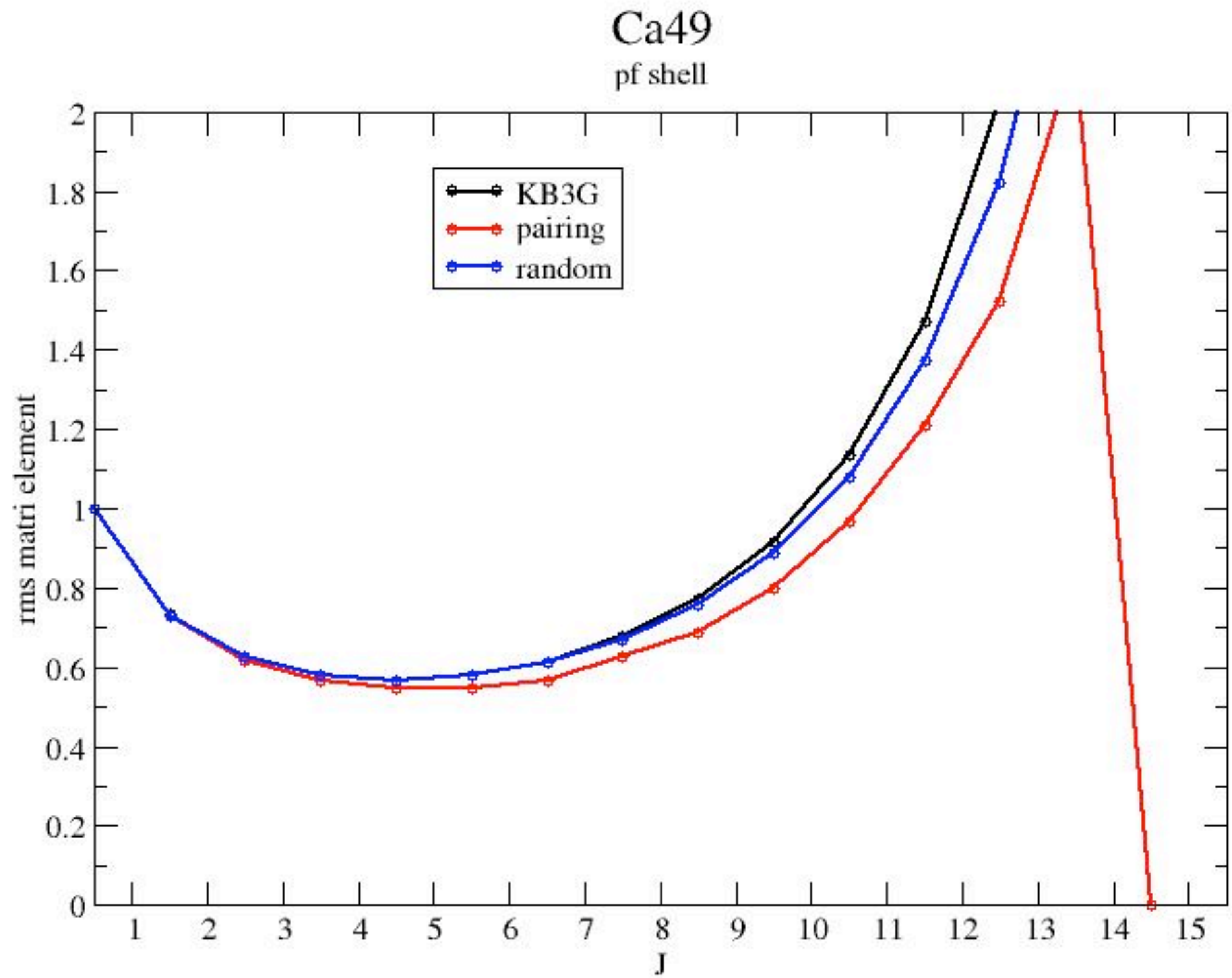
scaled, shifted centroids:

$$\left(d_J - d_0\right)/\sigma_{J=0}$$


Ca50
pf shell
KB3G
pairing
quadrupole-quadrupole
centroid (scaled)
J
0
5
10
15
20
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16



Ca 50, pf shell
random interactions
centroid (scaled and shifted)
10
5
0
-5
-10
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
J



Ca49
pf shell
KB3G
pairing
random
rms matri element
J
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15

What have we found?

The *rms matrix elements* as a function of J show remarkably universal behavior.

This behavior, aside from overall scale, is nearly independent of interaction.

It also appears remarkably insensitive to the model space.
The main constraint *appears* to be a cut-off in maximum J.

The *centroids* are much more interaction-dependent. However, they show surprisingly smooth behavior as one progresses in *J*.

Is there an even *simpler* model in which we can understand some of these trends?

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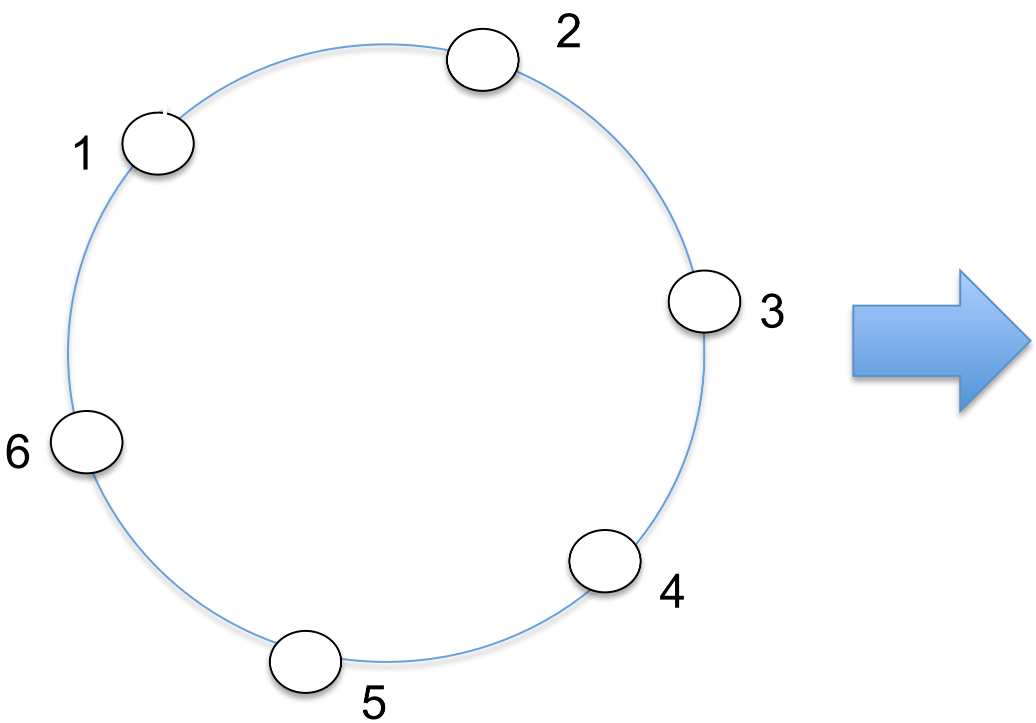
The origin of nuclear spin

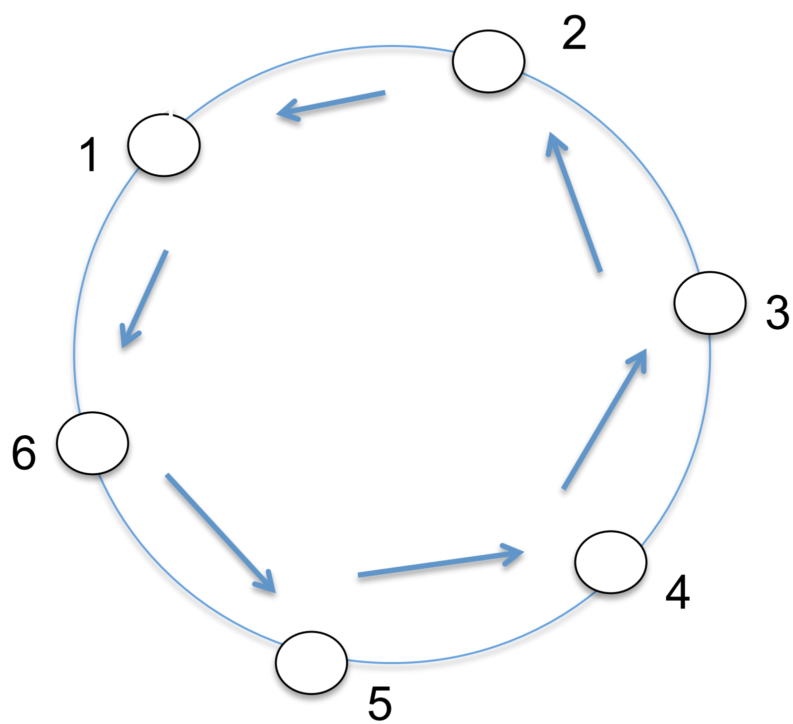
Can we go more abstract---
Can we impose a nontrivial symmetry on a random matrix?

Consider C_n symmetry:



$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{pmatrix}$$

The origin of nuclear spin



The generator of rotations is

$$T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The origin of nuclear spin

The generator of rotations is

$$T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The general matrix invariant under $H = T^{-1} H T$ is

$$H = \begin{pmatrix} a & b & c & d & c & b \\ b & a & b & c & d & c \\ c & b & a & b & c & d \\ d & c & b & a & b & c \\ c & d & c & b & a & b \\ b & c & d & c & b & a \end{pmatrix}$$

We can solve **H** by a Fourier transform; each eigenvalue is associated with a "quantum number" (momentum)

The general matrix invariant under H = T^{-1} H T is

$$h_m = \sum_k 2\cos\left(\frac{\pi mk}{N}\right) H_{1,1+k}$$

$$H = \begin{pmatrix} \boxed{a \quad b \quad c \quad d \quad c \quad b} \\ b & a & b & c & d & c \\ c & b & a & b & c & d \\ d & c & b & a & b & c \\ c & d & c & b & a & b \\ b & c & d & c & b & a \end{pmatrix}$$

(It's straightforward to also find the analytic eigenvectors—sines and cosines, as you'd imagine)

The origin of nuclear spin

While this is cute, can we do anything more?

What if we replace each entry by a random matrix?

$$H = \begin{pmatrix} a & b & c & d & c & b \\ b & a & b & c & d & c \\ c & b & a & b & c & d \\ d & c & b & a & b & c \\ c & d & c & b & a & b \\ b & c & d & c & b & a \end{pmatrix}$$

The origin of nuclear spin

While this is cute, can we do anything more?

What if we replace each entry by a random matrix?

(The dimensions of the submatrices represent internal degrees of freedom)

$$H = \begin{pmatrix} A & B & C & D & C & B \\ B & A & B & C & D & C \\ C & B & A & B & C & D \\ D & C & B & A & B & C \\ C & D & C & B & A & B \\ B & C & D & C & B & A \end{pmatrix}$$

We can not longer analytically solve the matrix, *but* we *can* project out matrices representing the irreps (irreducible representations) of the symmetry:

As before, we identify the submatrices with an index:

$$\mathbf{F}_0 = \mathbf{A},\ \mathbf{F}_1 = \mathbf{B},\ \mathbf{F}_2 = \mathbf{C}\ldots$$

$$h_m = \sum_k 2\cos\left(\frac{\pi m k}{N}\right) F_k$$

$$H = \begin{pmatrix} A & B & C & D & C & B \\ B & A & B & C & D & C \\ C & B & A & B & C & D \\ D & C & B & A & B & C \\ C & D & C & B & A & B \\ B & C & D & C & B & A \end{pmatrix}$$

....only now $\mathbf{h}_m$ is a *matrix*.

Better yet, we can compute the *width* of each $\mathbf{h}_m$

The origin of nuclear spin

$$h_m = \sum_k 2\cos\left(\frac{\pi m k}{N}\right) F_k$$

Assuming all the submatrices are independent...

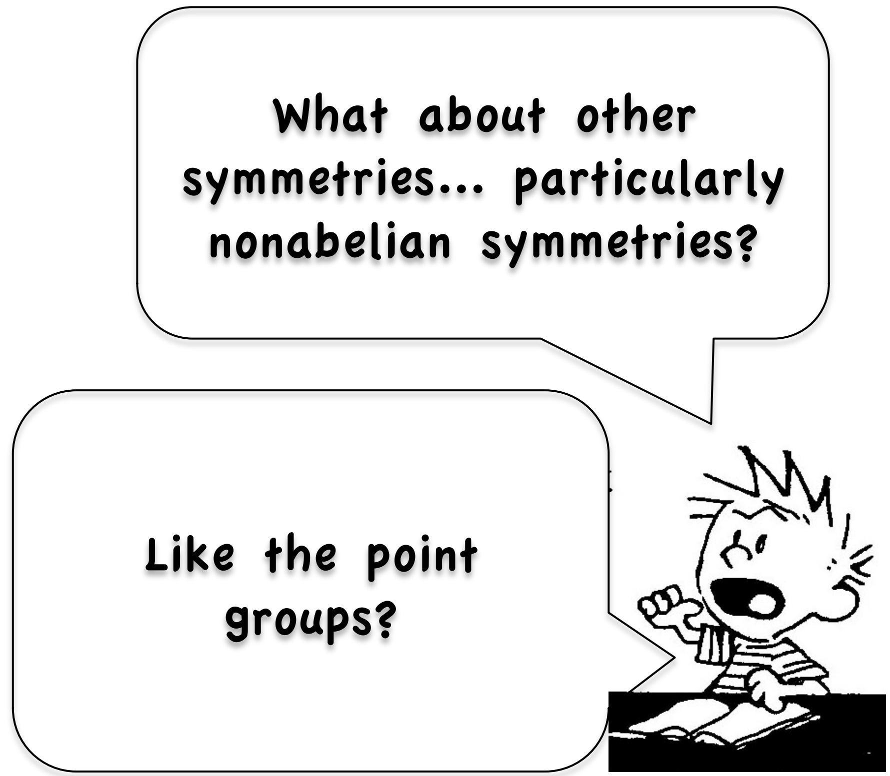
$$\sigma_m^2 = \sum_k 4\cos^2\left(\frac{\pi m k}{N}\right)\sigma^2(F_k)$$

Assuming all the submatrices have the same width...

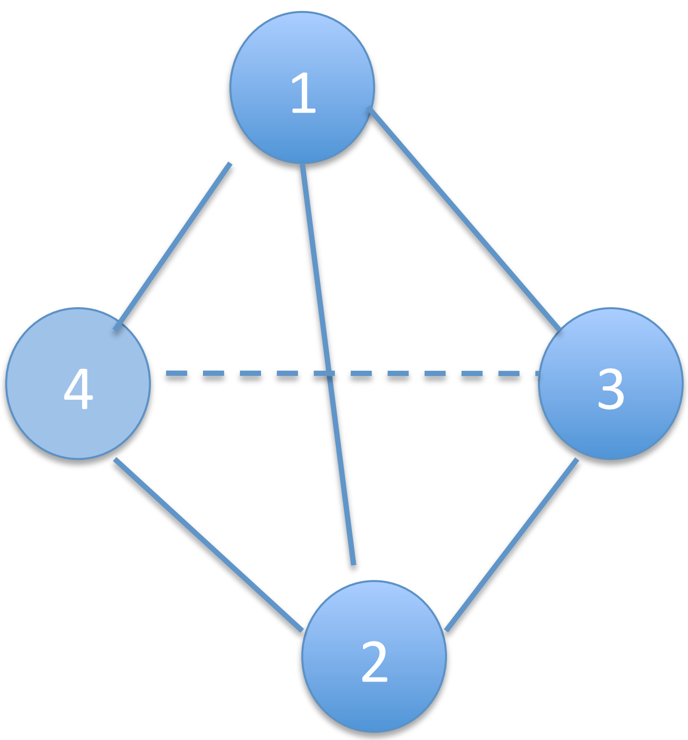
$$\sigma_m^2 = \sum_k 4\cos^2\left(\frac{\pi m k}{N}\right)\sigma^2$$

$$\boxed{\approx 2\sigma^2\left(1+\delta_{m,0}\right)}$$

The origin of nuclear spin



The Tetrahedron



$$H = \begin{pmatrix} A & B & B & B \\ B & A & B & B \\ B & B & A & B \\ B & B & B & A \end{pmatrix}$$

One-dimensional irrep:
(most symmetric)

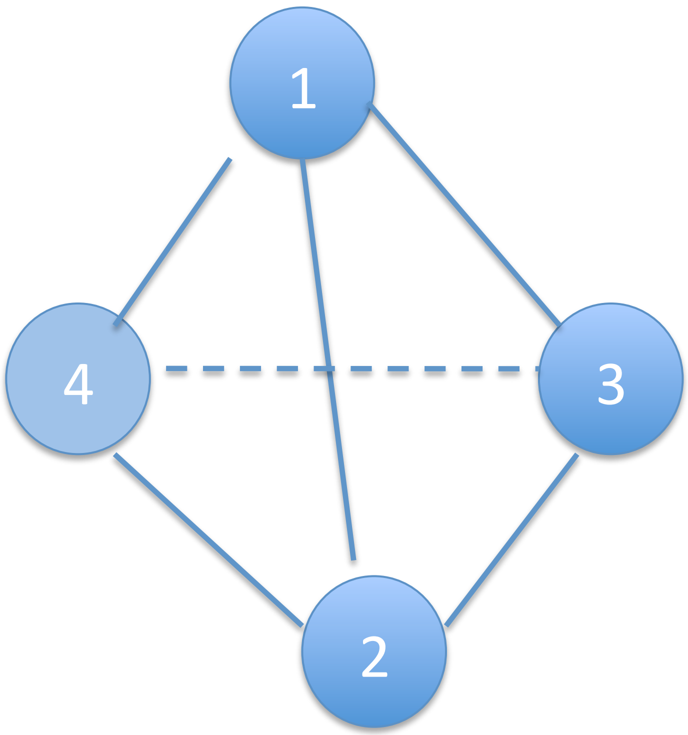
$\boxed{\mathbf{h= A+3B} \qquad \sigma^2{}_1 = 10}$

Largest width
so most likely
ground state

3-dimensional irrep:

h = A – B $\qquad \sigma^2{}_3 = 2$

The Tetrahedron



Transformed so block-diagonal in irreps

$$H' = \begin{pmatrix} A+3B & 0 & 0 & 0 \\ 0 & A-B & 0 & 0 \\ 0 & 0 & A-B & 0 \\ 0 & 0 & 0 & A-B \end{pmatrix}$$

One-dimensional irrep:
(most symmetric)

$\boxed{\mathbf{h= A+3B} \qquad \sigma^2{}_1 = 10}$

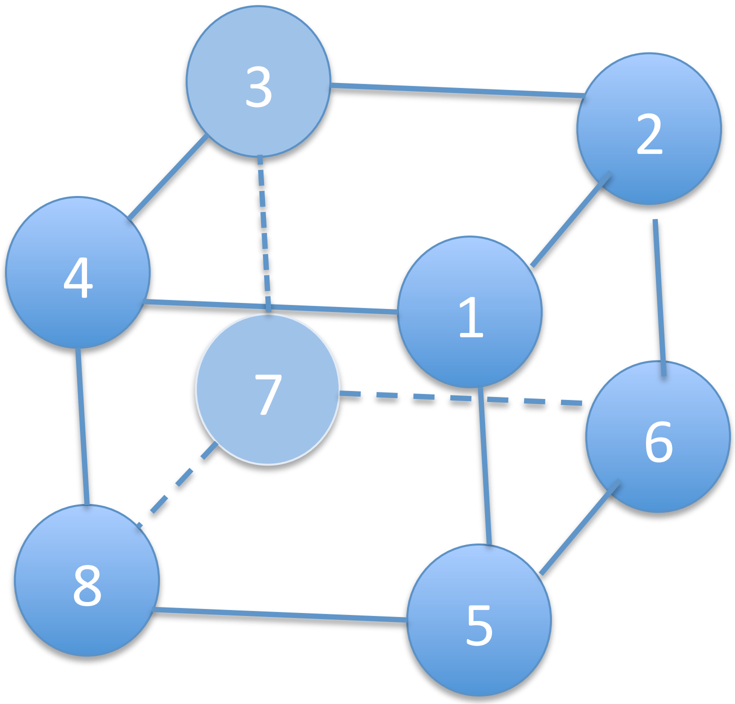
Largest width
so most likely
ground state

3-dimensional irrep:

$\mathbf{h = A - B} \qquad \sigma^2{}_3 = 2$

The origin of nuclear spin

The Cube



$$H = \begin{pmatrix} A & B & C & B & B & C & D & C \\ B & A & B & C & C & B & C & D \\ C & B & A & B & D & C & B & C \\ B & C & B & A & C & D & C & B \\ B & C & D & C & A & B & C & B \\ C & B & C & D & B & A & B & C \\ D & C & B & C & C & B & A & B \\ C & D & C & B & B & C & B & A \end{pmatrix}$$

One-dimensional irreps:
(most symmetric)

$$\boxed{\mathbf{h = A \pm 3B + 3C \pm D} \quad \sigma^2_1 = 20}$$

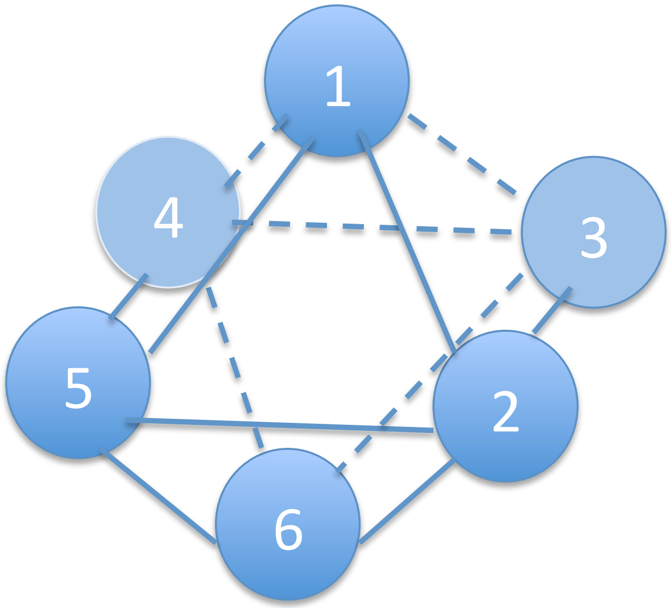
Largest width
so most likely
ground state

3-dimensional irreps:

$$\mathbf{h = A\text{-}C \pm (B\text{-}D)} \quad \sigma^2_3 = 4$$

The origin of nuclear spin

The Octahedron



$$H = \begin{pmatrix} A & B & C & B & B & B \\ B & A & B & C & B & B \\ C & B & A & B & B & B \\ B & C & B & A & B & B \\ B & B & B & B & A & C \\ B & B & B & B & C & A \end{pmatrix}$$

One-dimensional irrep:
(most symmetric)

$\boxed{\mathbf{h = A+4B+C} \qquad \sigma^2_1 = 18}$

Largest width so most likely ground state

2-dimensional irrep:

h = A-2B+C $\qquad \sigma^2_2 = 6$

3-dimensional irrep:

h = A-C $\qquad \sigma^2_3 = 2$

What have we learned so far?

If we impose symmetries on a random matrix (leaving additional degrees of freedom)....

... the lowest dimension / "most symmetric" irreps have largest widths and thus dominate the ground state

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What about *continuous* symmetries?
Like rotation?

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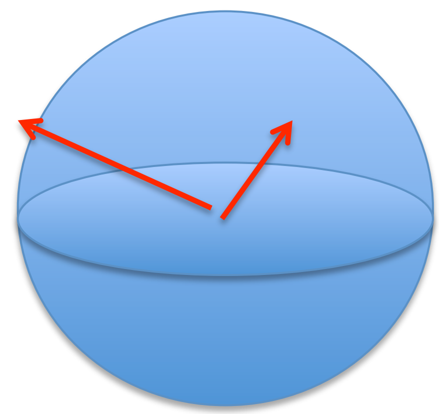
Starting from a rotationally invariant Hamiltonian:

$$H(\theta'\phi',\theta\phi) = F(\omega)$$

$$\cos\omega = \cos\theta'\cos\theta + \sin\theta'\sin\theta\cos(\phi'-\phi)$$



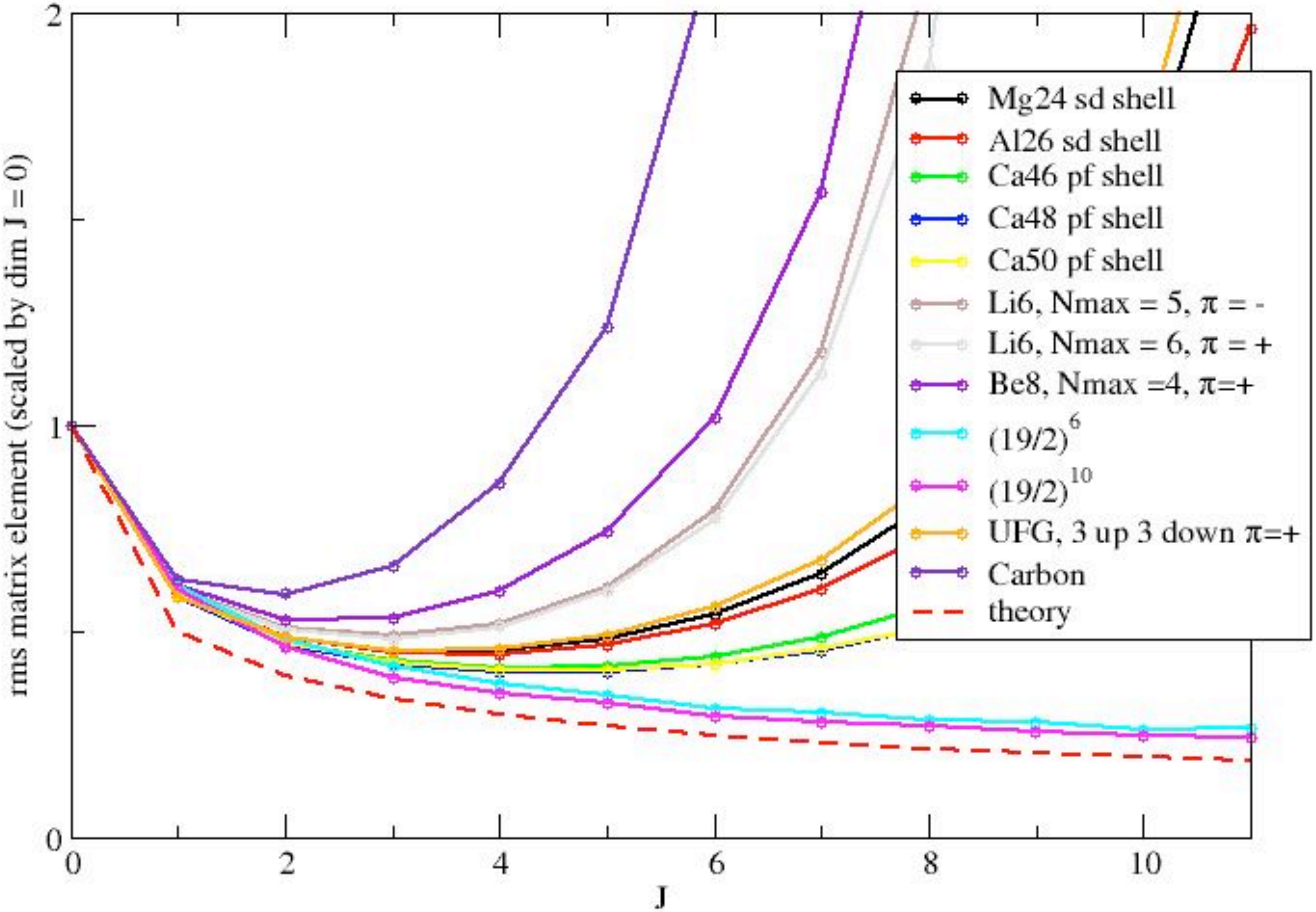
....we can project out Hamiltonians with good L:

$$H_L = 2\pi\int_0^\pi P_L(\cos\omega)F(\omega)d\cos\omega$$

From this we can compute the width as a function of L:

$$\sigma_L^2 = 4\pi^2\int_0^\pi P_L^2(\cos\omega)\sin^2\omega d\omega$$

For L= 0, 1,2,3, 4 values: 1.571, 0.393, 0.245, 0.178, 0.139



rms matrix element (scaled by dim J = 0)
J
0
1
2
0
2
4
6
8
10
Mg24 sd shell
Al26 sd shell
Ca46 pf shell
Ca48 pf shell
Ca50 pf shell
Li6, Nmax = 5, π = -
Li6, Nmax = 6, π = +
Be8, Nmax =4, π=+
(19/2)^6
(19/2)^10
UFG, 3 up 3 down π=+
Carbon
theory

Summary:

Why do even-even nuclei have $J = 0$?

The force plays a role, but **numerical simulations** suggest *most* forces would produce a *qualitatively* similar result.

I find the rms matrix element has near-universal behavior as a function of J.

By considering random matrices with symmetries, we find that the ground state is dominated by lowest-dimension / most symmetric irrep.

Work to be done:

Need to formalize results from points groups.

Make application to continuous symmetries more rigorous.

Can I better motivate mapping/modeling of many-body systems?

What about other phenomena? Such as R_{62}/ R_{42} ratio?
(Preliminary results suggest a strong correlation; furthermore the equivalent R_{12}/R_{42} has no correlation – a prediction!)

What about other symmetries (SU(3)) and broken symmetries?

A lot of fun work ahead!

Mapping onto many-body simulations is not trivial:

-- Different *J* spaces have different dimensions

-- Level densities is Gaussian, not GOE

To account for this, choose Gaussian with width

$$\sigma_L^2(eff) = \sqrt{N_L}\,\sigma_L^2$$

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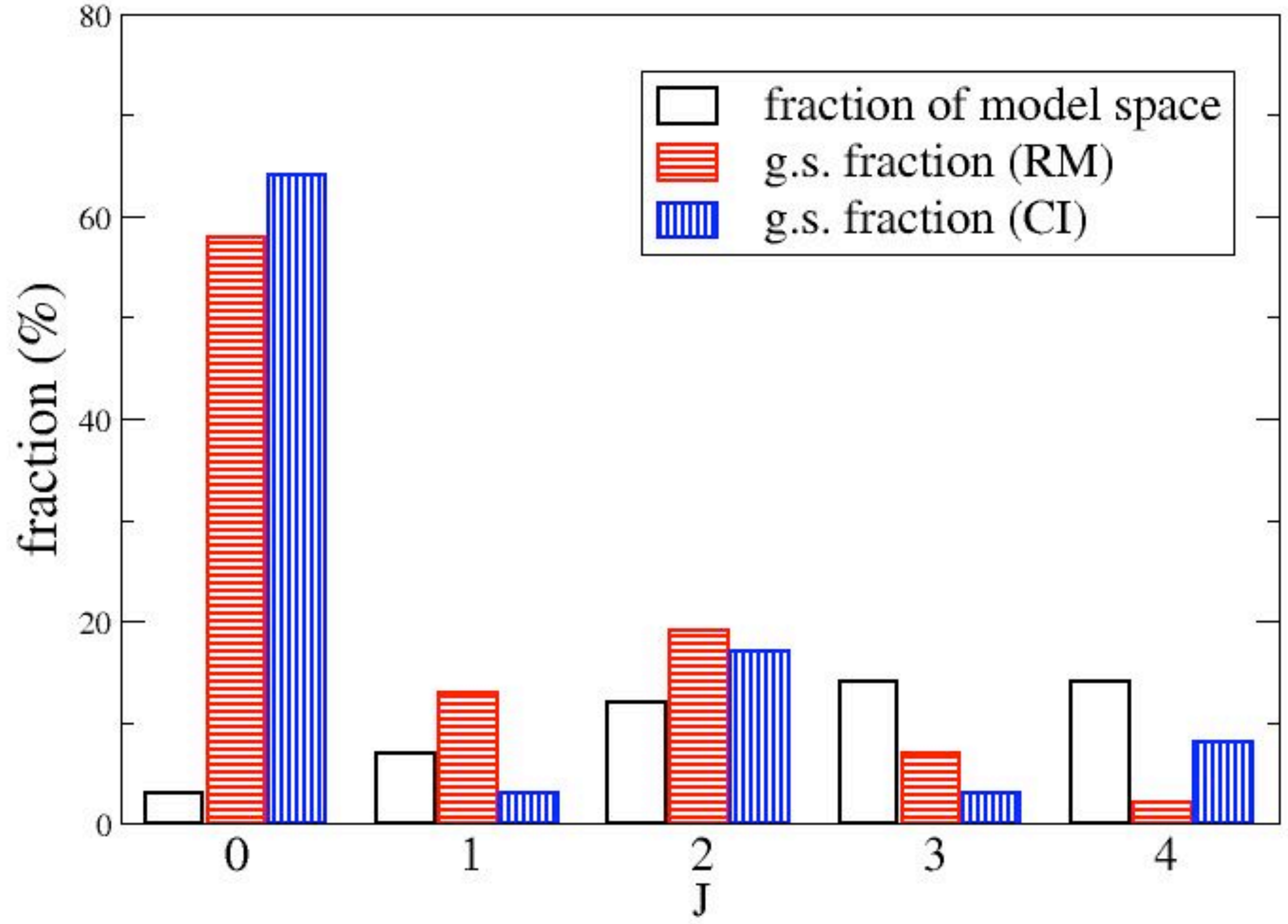

The origin of nuclear spin

“single-*j* shell: $(21/2)^8$

J	f_{space} (%)	f_{RM}	f_{CI} (%)
0	0.4	33	55
1	0.5	0.2	0
2	1	9	7
3	1	3	0.2
4	2	11	2

IBM, N = 7

J	f_{space} (%)	f_{RM}	f_{CI} (%)
0	11	81	55
1	N/A	-	-
2	17	14	13
3	6	0.1	0.08
4	17	4	4