Yrast states in two-component Bose gases

Thomas Papenbrock

and OAK RIDGE NATIONAL LABORATORY

Happy Birthday, George!

Bertschfest Seattle, September 7-9 2012

Research partly funded by the US Department of Energy and the Alexander von Humboldt foundation

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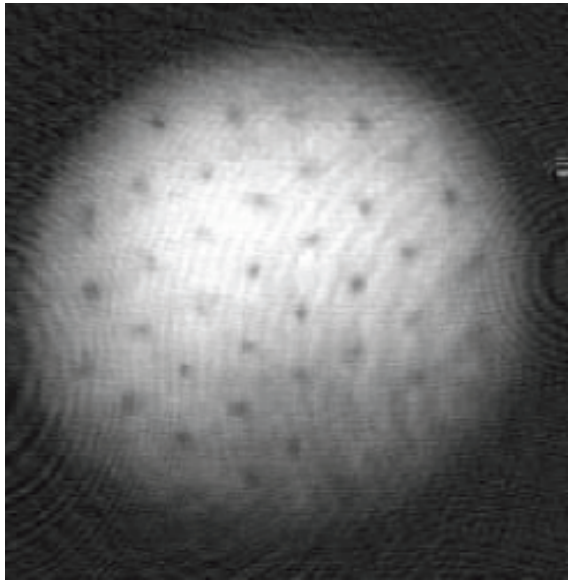

Abo-Shaeer et al.,
Science 292, 476 (2001)

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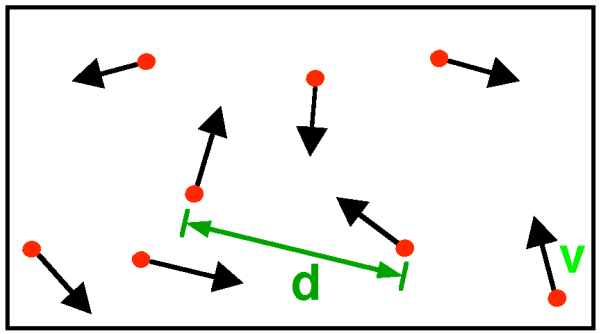
Bose-Einstein condensation

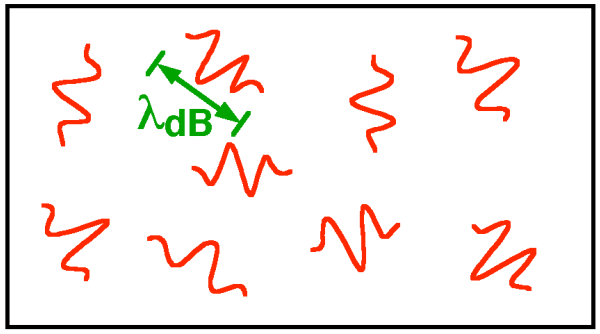


High Temperature T:

thermal velocity v

density d^{-3}

"Billiard balls"

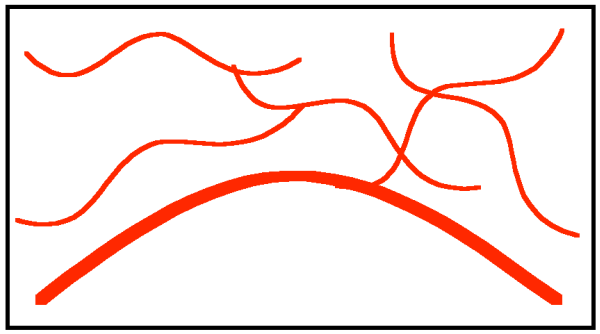


Low Temperature T:

De Broglie wavelength

$\lambda_{dB}=h/mv \propto T^{-1/2}$

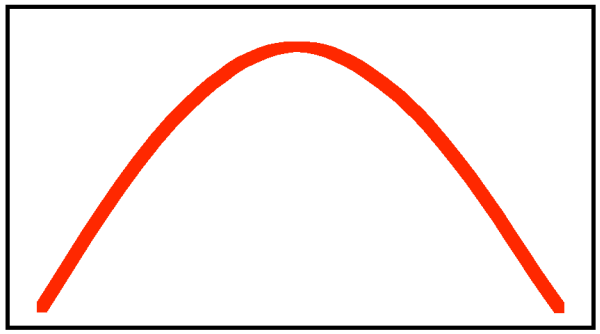
"Wave packets"



$T=T_{crit}$: Bose-Einstein Condensation

$\lambda_{dB} \approx d$

"Matter wave overlap"

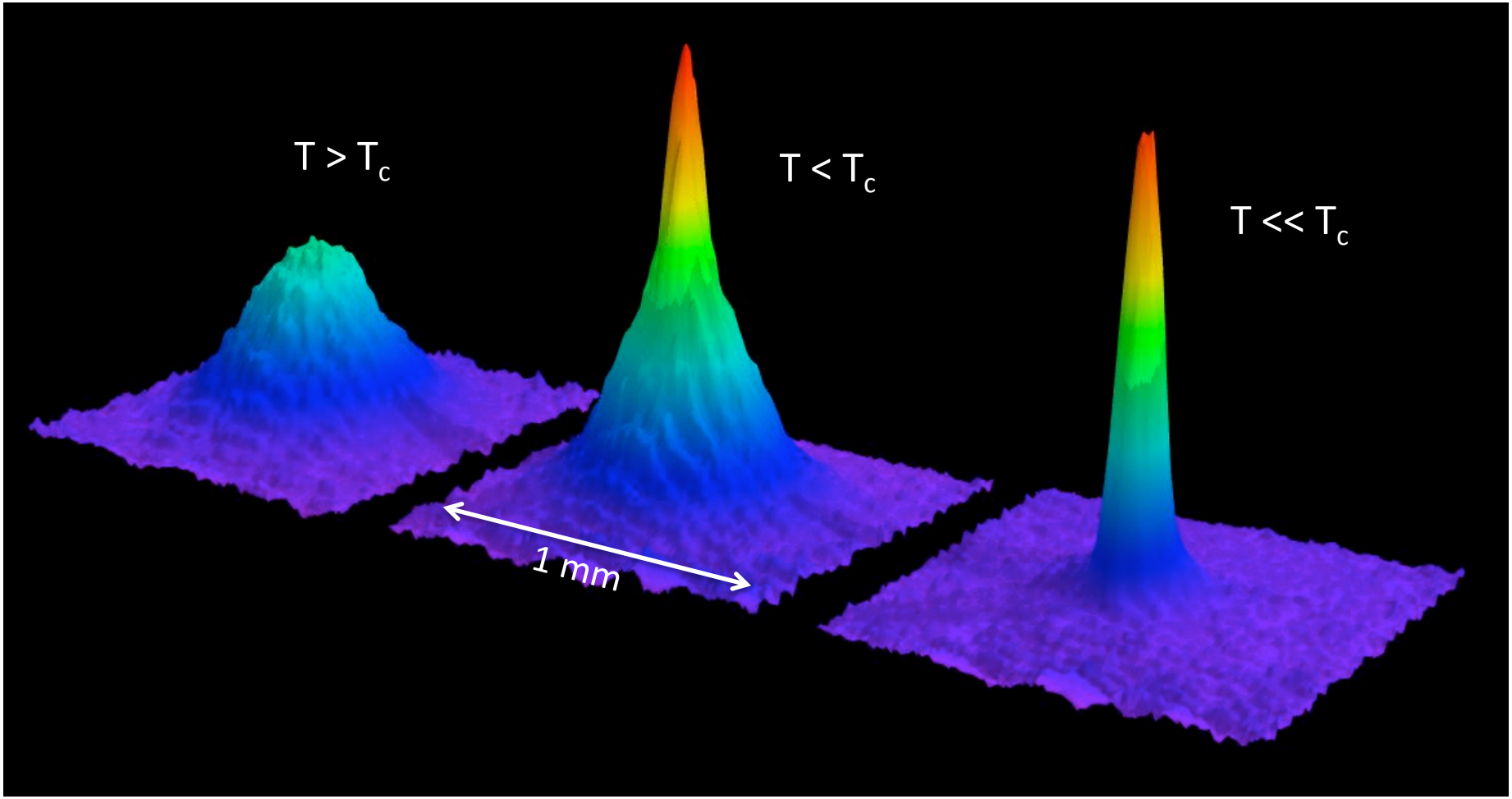


T=0: Pure Bose condensate

"Giant matter wave"

W. Ketterle (2003)

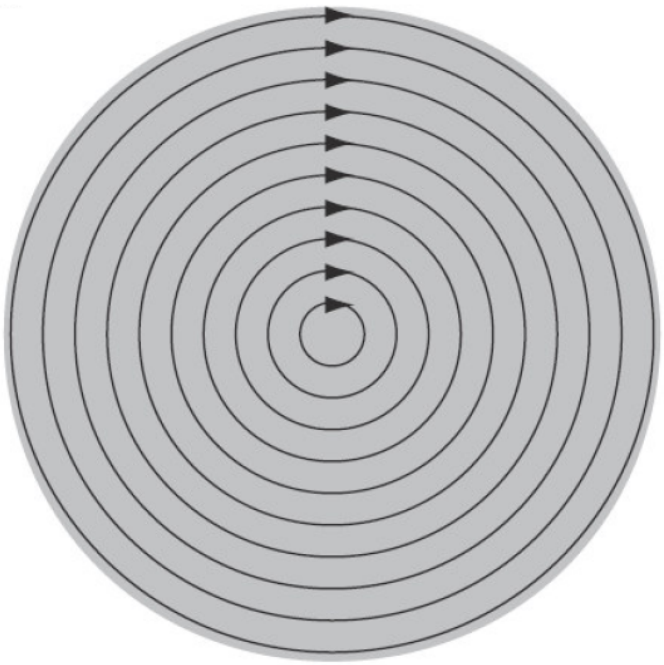
Formation of a Bose-Einstein condensate



Bose-Einstein condensates are macroscopic quantum objects (picture taken after 6ms of expansion), [Ketterle's MIT group 1995]

Normal fluids under rotation

Normal fluid in (slowly) rotating bucket



Fast stirred beverage

"Rotational" flow

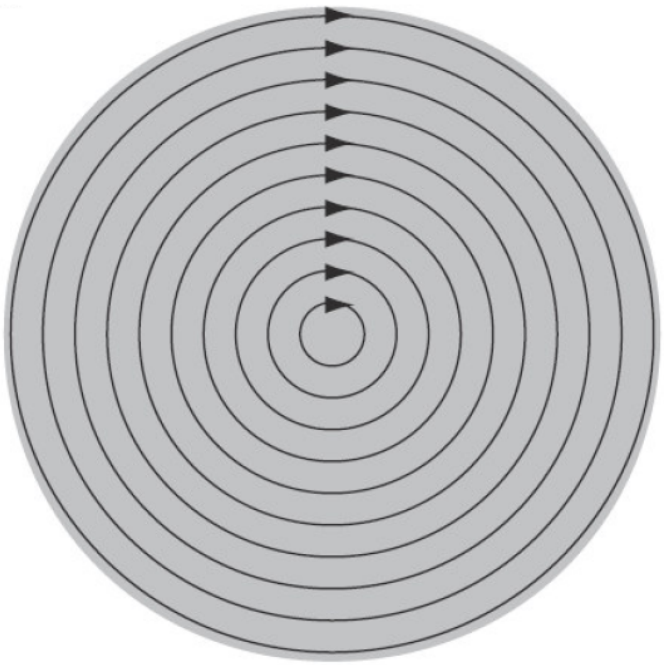
$$\vec{v} = \vec{\Omega} \times \vec{r}$$
$$\vec{\nabla} \times \vec{v} = 2\vec{\Omega}$$

"Irrotational" flow

$$\vec{v} \propto \frac{1}{r}\vec{e}_\phi$$
$$\vec{\nabla} \times \vec{v} = 0$$

Normal fluids under rotation

Normal fluid in (slowly) rotating bucket



Fast stirred beverage

"Rotational" flow

$$\begin{aligned} \vec{v} &= \vec{\Omega} \times \vec{r} \\ \vec{\nabla} \times \vec{v} &= 2\vec{\Omega} \end{aligned}$$

"Irrotational" flow

$$\begin{aligned} \vec{v} &\propto \frac{1}{r}\vec{e}_\phi \\ \vec{\nabla} \times \vec{v} &= 0 \end{aligned}$$

Bose-Einstein condensate under rotation

BEC wave function

$$\psi(\vec{r}) = |\psi(\vec{r})| e^{iS(\vec{r})}$$

Implies irrotational flow

$$\begin{aligned} \vec{v} &= \vec{\nabla} S \\ \vec{\nabla} \times \vec{v} &= 0 \end{aligned}$$

However: wave function is single-valued, and the phase circulation is quantized

$$\oint d\vec{l} \cdot \vec{v} = \oint d\vec{l} \cdot \vec{\nabla} S = 2\pi n \ , \quad n = 0,\ 1,\ 2,\ \ldots$$

Stokes theorem implies that a singularity (vortex) is present

$$2\pi n = \oint d\vec{l} \cdot \vec{v} = \int d\vec{A} \cdot \vec{\nabla} \times v$$

$$\left(\vec{\nabla} \times \frac{1}{r} \vec{e}_\phi = 2\pi \delta^{(2)}(\vec{r}) \vec{e}_z \right)$$

Rotating gases in harmonic traps

Almost all experiments are in the Thomas-Fermi regime, where the interaction energy dominates the kinetic energy [K. Kasamatsu, M. Tsubota & M. Ueda, Int. J. Mod. Phys. B 19, 1835 (2005)]

$$\frac{\hbar^2 a}{m}\rho \gg \hbar\omega$$

We are interested in the opposite regime of perturbatively weak interactions ("quantum Hall regime")

$$\frac{\hbar^2 a}{m}\rho \ll \hbar\omega$$

As we will see, this is 1st order perturbation theory of a highly degenerate level.

Yrast states in rotating Bose gases

N bosons in a harmonic trap
L units of angular momentum, take L_z=L maximally aligned state
Yrast line: all energy is in rotational motion – no radial excitations

→Single-particle states are $\psi_m = (x+iy)^m e^{-½(x^*x+y^*y+z^*z)}$ denoted as $\psi_m = z^m$

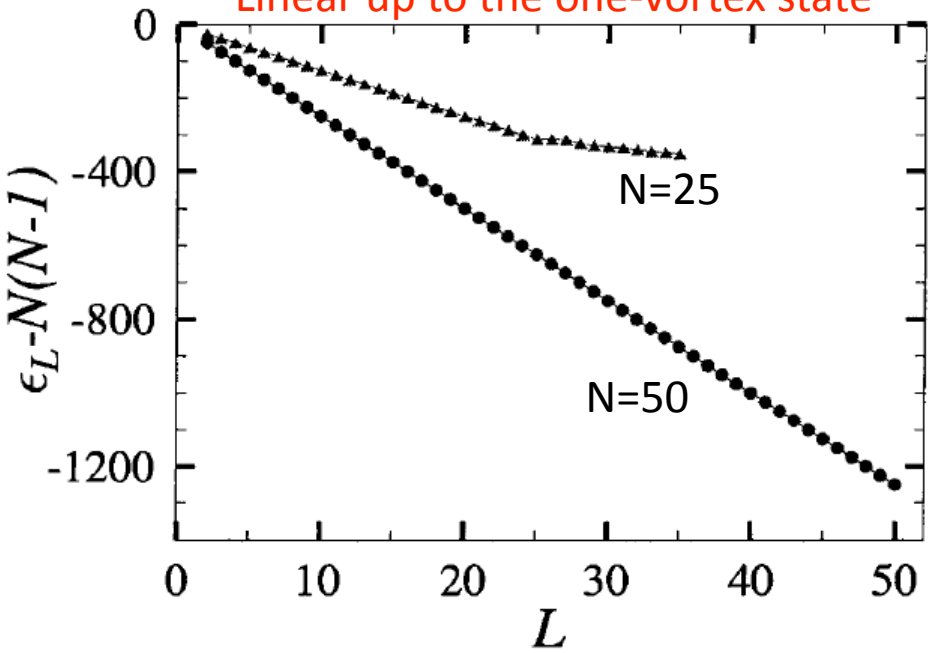
Hilbert space is partition space!
All partitions of L into at most N integers
Example: L=4: (4); (3,1); (2,2); (2,1,1); (1,1,1,1)
Dimension $\sim e^{\sqrt{L}}$

In the absence of interactions, all states are degenerate, with energy $E=L\hbar\omega$

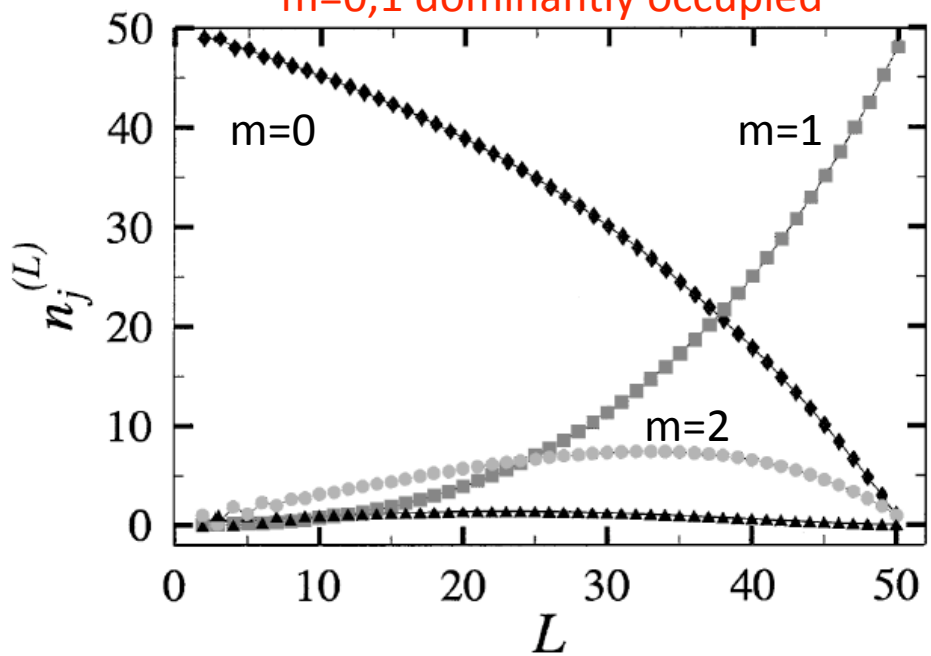
How will weak repulsive interactions lift this degeneracy?
[B. Mottelson, Phys. Rev. Lett. 83, 2695 (1999)]

Exact diagonalizations yielded simple results…

Interaction energy
Linear up to the one-vortex state



Occupation numbers
m=0,1 dominantly occupied



… and a conjecture

$$|L\rangle = \mathcal{N} \sum_{p_1<p_2<\ldots<p_L} (z_{p_1} - z_c)(z_{p_2} - z_c)\ldots(z_{p_L} - z_c)\,|0\rangle$$

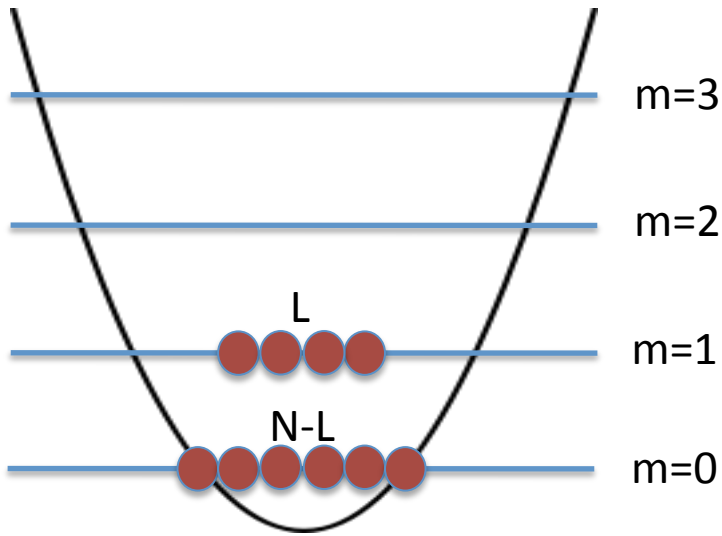
$$z_c = N^{-1} \sum_{p=1}^{N} z_p \qquad \text{(center of mass)}$$

Bertsch and TP, Phys. Rev. Lett. 83, 5412 (1999)

Understanding the Yrast states

$$|L\rangle = \mathcal{N} \sum_{p_1<p_2<\ldots<p_L} (z_{p_1} - z_c)(z_{p_2} - z_c)\ldots(z_{p_L} - z_c)\,|0\rangle$$

Mean-field picture [Kavoulakis, Mottelson, and Pethick, Phys. Rev. A 62, 063605 (2000)]:
L particles carry one unit of angular momentum each; remaining particles in ground state.
m=0,1 single-particle states have dominant occupations.



$$\psi_L = \sum_{1\le i_1<i_2\ldots<i_L\le N} z_{i_1} z_{i_2} \cdots z_{i_L}$$

Center-of-mass insertions make state ``translationally invariant'' (angular momentum of center of mass is conserved)

Single-vortex state (L=N)
[Wilkin, Gunn, & Smith, Phys. Rev. Lett. 80, 2265 (1998)]

$$\psi_{L=N}^{\text{exact}} = \prod_{i=1}^{N} [z_i - z_c]$$

Properties of exact solution

Analytical proofs:

- Eigenstate for contact interaction: [Smith and Wilkin, Phys. Rev. A 62, 061602 (2000)]
- Eigenstate for any interaction: [TP and Bertsch, J. Phys. A 34, 603 (2001)]
- Ground state for large set of repulsive interactions: [M. S. Hussein and O. K. Vorov, Physica 312 B, 550 (2002)]

Yrast states are universal

Subspace in Hilbert space with dimension L+1 is invariant for any two-body Hamiltonian

$$e_\lambda(z_1, \ldots, z_N) = \sum_{1 \leqslant p_1 < p_2 < \cdots < p_\lambda \leqslant N} z_{p_1} z_{p_2} \ldots z_{p_\lambda}$$

$$\mathcal{V}_1 = \mathrm{span}\{e_{L-\lambda}\, e_1^\lambda : \lambda = 0, 1, 2, \ldots, L-2\}$$

$$e_L(z_1 - z_c, \ldots, z_N - z_c) = \sum_{m=0}^{L} \binom{N-m}{L-m} (-z_c)^{L-m} e_m(z_1, \ldots, z_N)$$

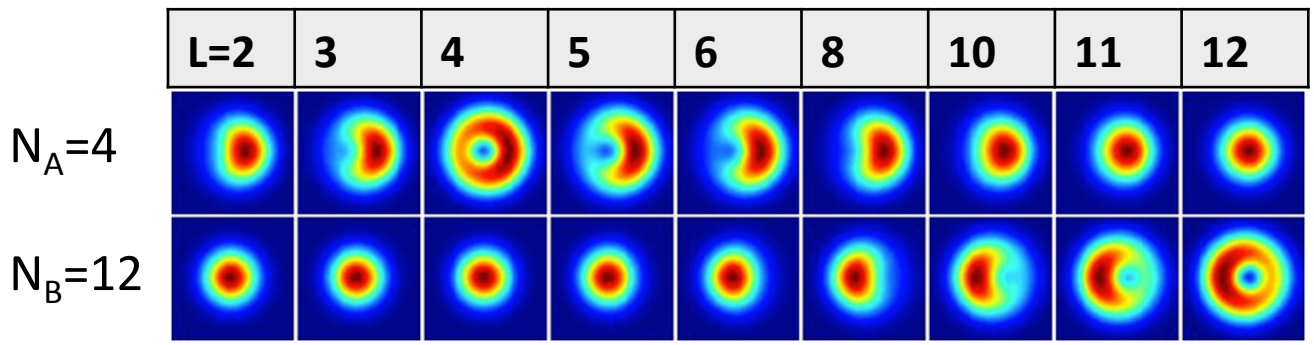
Mixtures of Bose Gases under Rotation

[Bargi, Christensson, Kavoulakis, and Reimann, Phys. Rev. Lett. 98, 130403 (2007)]

Question: What are the rotational properties of two-component BECs in harmonic traps?

Main results (assume ``isospin symmetric'' interactions):

1. For unequal number of bosons, coreless vortices are formed. One species fills the vortex core and remains non-rotating, whereas the other species rotates.
2. Analytical expressions (from inspection of numerical results) for dispersion relation and occupation numbers.

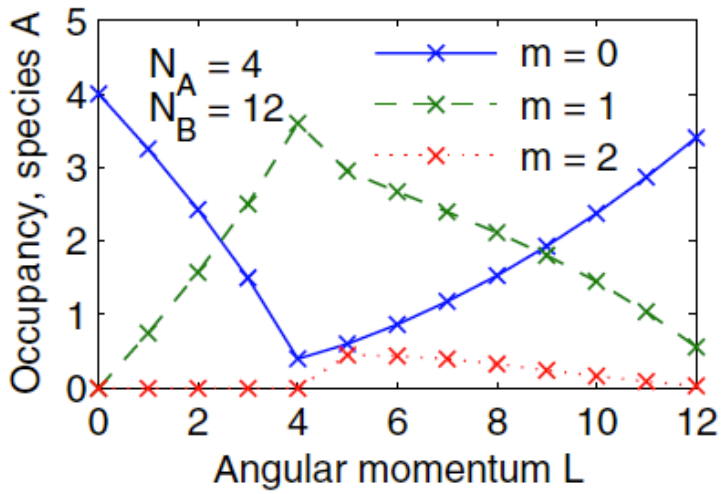


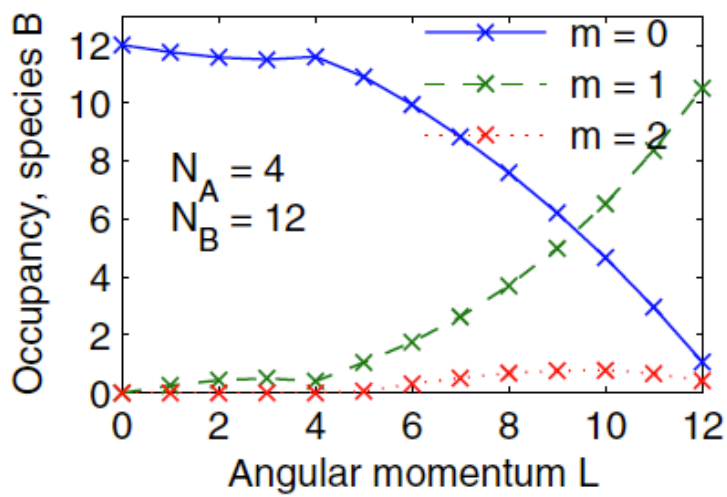
$$\mathcal{E}_0(L)/v_0 = \tfrac{1}{2}N(N-1) - \tfrac{1}{2}NL + \tfrac{1}{4}L(L-1) \qquad 0 \le L \le N_{\min}$$

$$\mathcal{E}_0(L)/v_0 = \tfrac{1}{2}N(N-1) - \tfrac{1}{4}N_{\min}N - \tfrac{1}{4}NL + \tfrac{1}{4}N_{\min}(N_{\min}-1). \qquad N_{\min} \le L \le N_{\max}$$

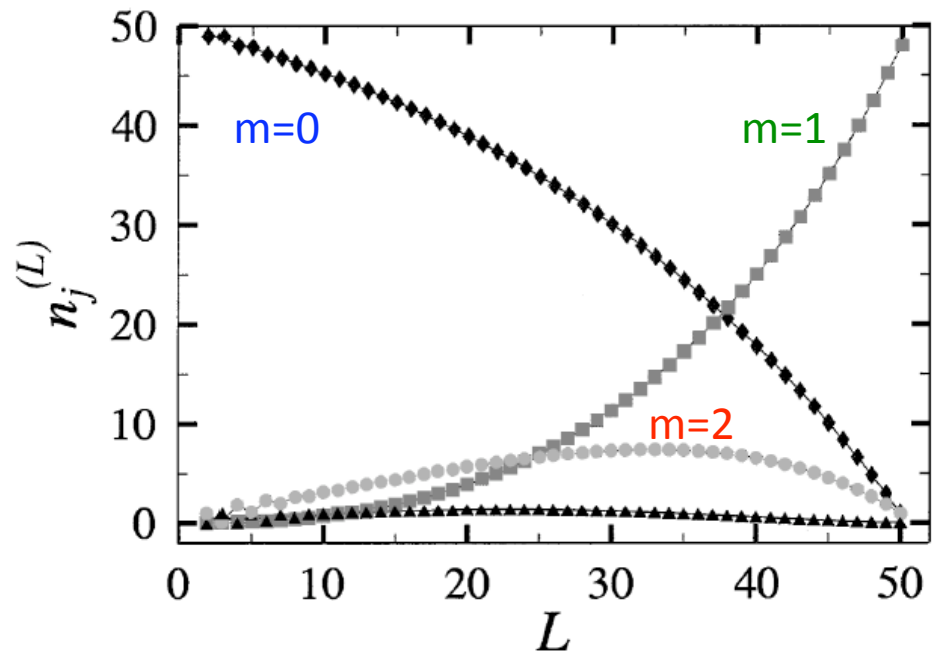
Occupation numbers

Occupations for two components





Occupations for single-component gas with N=50 bosons



Déjà vu!

Dominance of m=0,1 states

Constructing the solution

[TP, Reimann & Kavoulakis, Phys. Rev. Lett. 108, 075304 (2012)]

Hund's rule: repulsive interactions yield antisymmetric wave functions in position space
→Bosons will have antisymmetric isospin wave function T=0

States with m=0,1 are dominant
→P-wave pairs with isospin T=0

$$\hat{B}^\dagger \equiv \frac{1}{\sqrt{2}}(\hat{b}^\dagger_{1\downarrow}\hat{b}^\dagger_{0\uparrow} - \hat{b}^\dagger_{0\downarrow}\hat{b}^\dagger_{1\uparrow})$$

Many-boson state (read from right to left)

$$|\chi_\tau\rangle \equiv (\hat{T}_-)^{N-\tau}(\hat{b}^\dagger_{0\uparrow})^{A-L-\tau}(\hat{b}^\dagger_{1\uparrow})^{L-\tau}(\hat{B}^\dagger)^\tau|0\rangle$$

Get N↓, M↑

Make particle number A=N+M; have A-τ bosons ↑ and τ bosons ↓

L units of angular momentum

τ pairs

L+τ bosons total

Analytical approach to energies

Invariant subspace

$$\mathcal{M} \equiv \{R^n e_\lambda(z) e_{L-\lambda-n}(w)\} \qquad 0 \le \lambda \le \min(L, N), \quad 0 \le n \le L - \lambda$$

with center of mass

$$R \equiv \tfrac{1}{A}[e_1(z) + e_1(w)]$$

Ansatz for wave function

$$\psi_{L,n} = \sum_{\lambda=0}^{\min(L,N)} c_\lambda^{(n)} e_\lambda(z) e_{L-\lambda}(w)$$

yields energies

$$E_n = A(A-1)\frac{v_0}{2} + ALv_1 + 2v_2\varepsilon_n$$

$$\varepsilon_n = AL - n(A+1-n) \qquad n = 0, 1, 2, \ldots, \min(L, N)$$

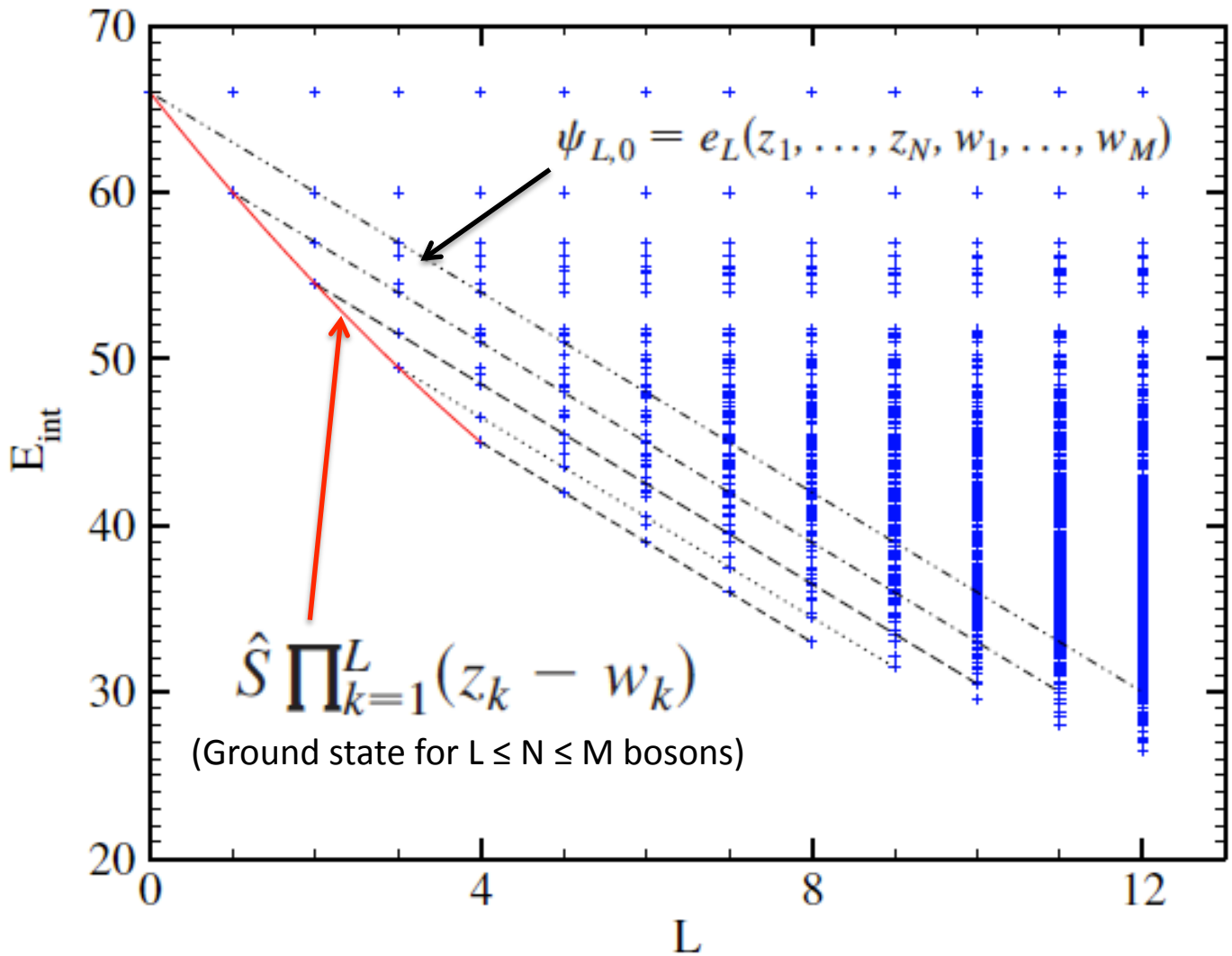
Matrix elements for contact …

$$v_m = (-1/2)^m/m!$$

… and for Coulomb interaction

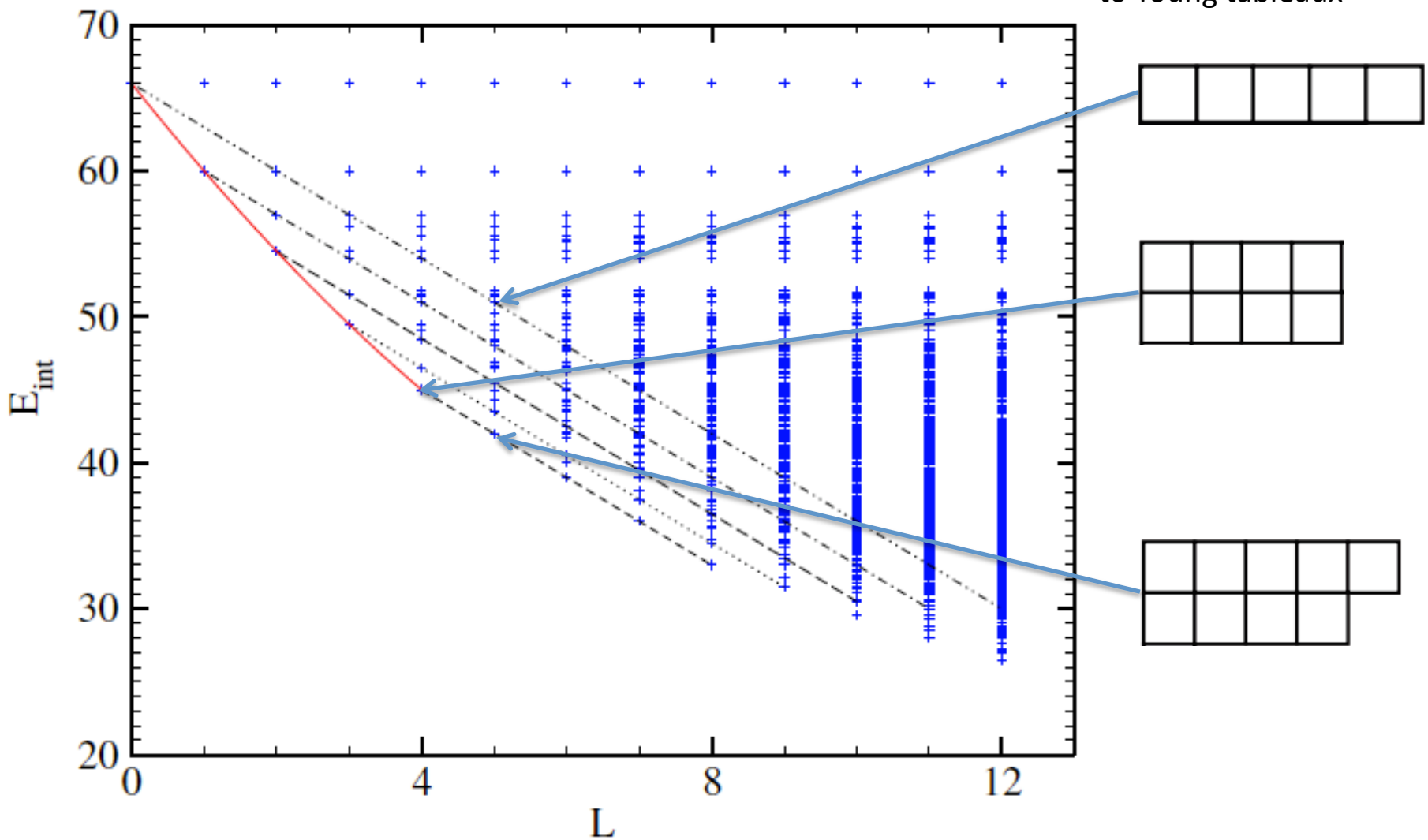
$$v_0 = \sqrt{\pi/2}, \quad v_1 = -v_0/4, \text{ and } v_2 = 3v_0/64$$

Spectrum (for contact interaction and N=4, M=8)



Spectrum (for contact interaction and N=4, M=8)

Wave functions correspond to Young tableaux



Summary

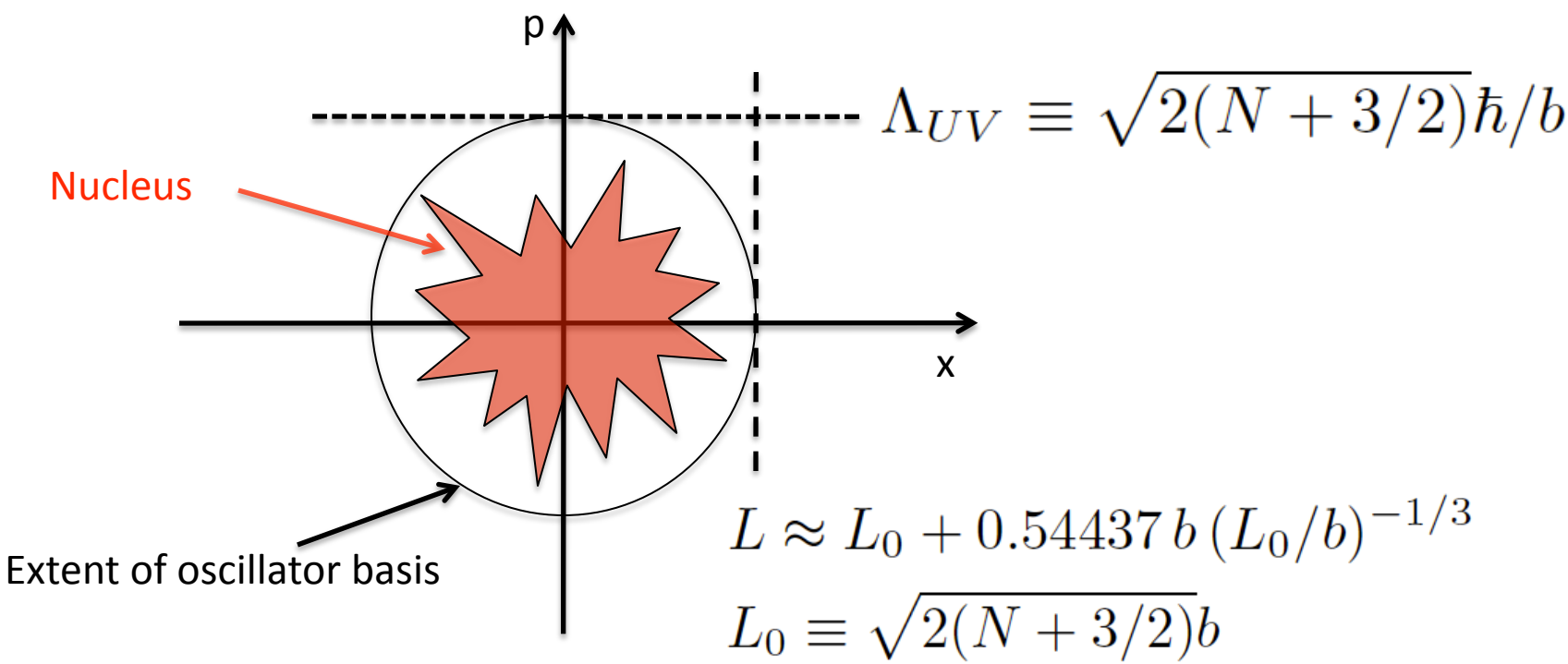
Yrast states for isospin-symmetric two-component Bose gases with weak interactions

- m=0, 1 orbitals dominate occupations
- Apply Hund's rule to bosons: ground state consists of p-wave pairs of bosons
- Exact solution (again!) due to small invariant subspace
- Solutions are universal for repulsive interactions (and lowest-energy states for contact, Coulomb, and dipolar forces)

Corrections to nuclear energies and radii in finite oscillator spaces

Calculations are performed in finite oscillator spaces. How can one reliably extrapolate to infinity?

Main idea: Look at phase space. Convergence in momentum space faster (super exponential) than in position space ("only" exponential tails)



Nucleus needs to "fit" into basis:

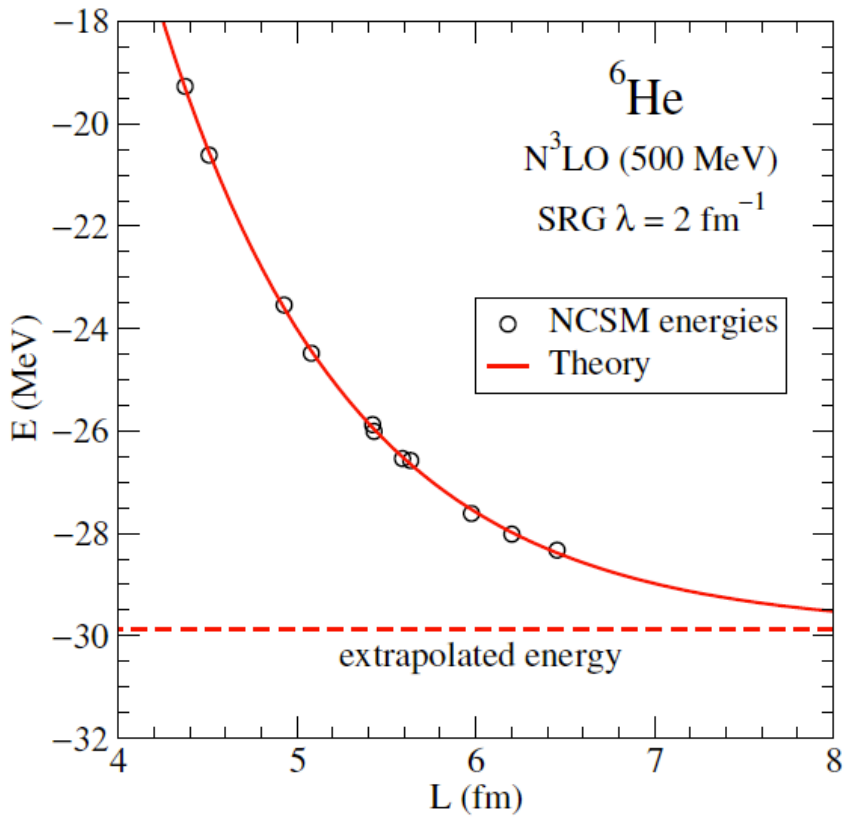
- Nuclear radius R < L
- cutoff of interaction $\Lambda < \Lambda_{UV}$

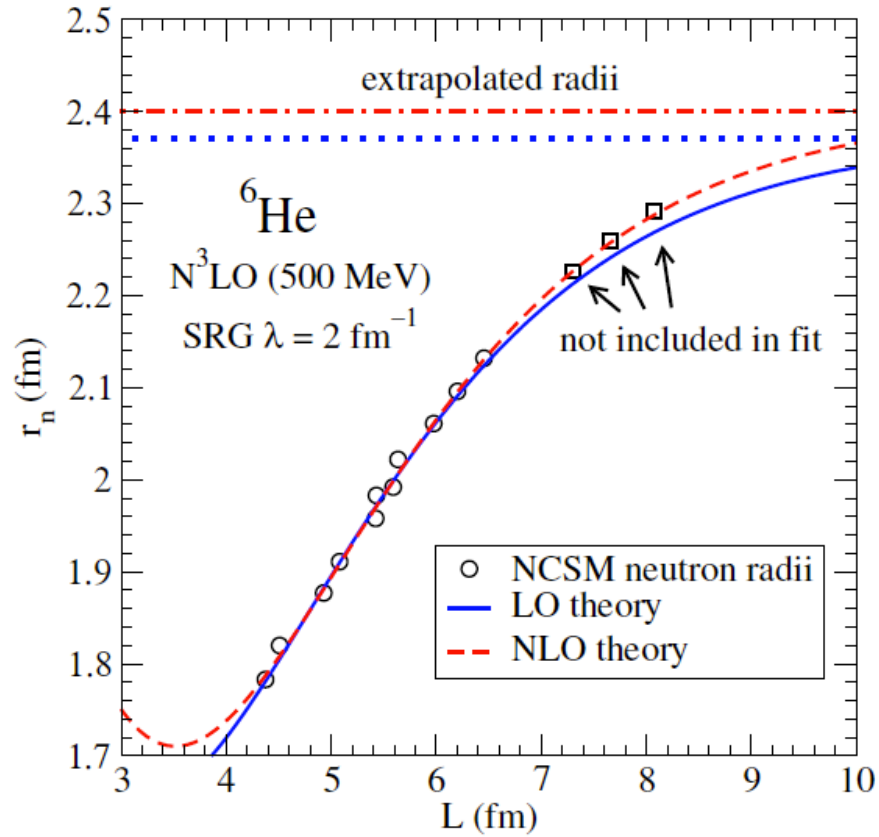
Extrapolations at work

- UV fully converged (convergence is quick once $\Lambda < \Lambda_{UV}$ due to super exponential behavior)
- IR convergence is slower due to exponential decay of wave function
- Dirichlet boundary condition at x=L in position space

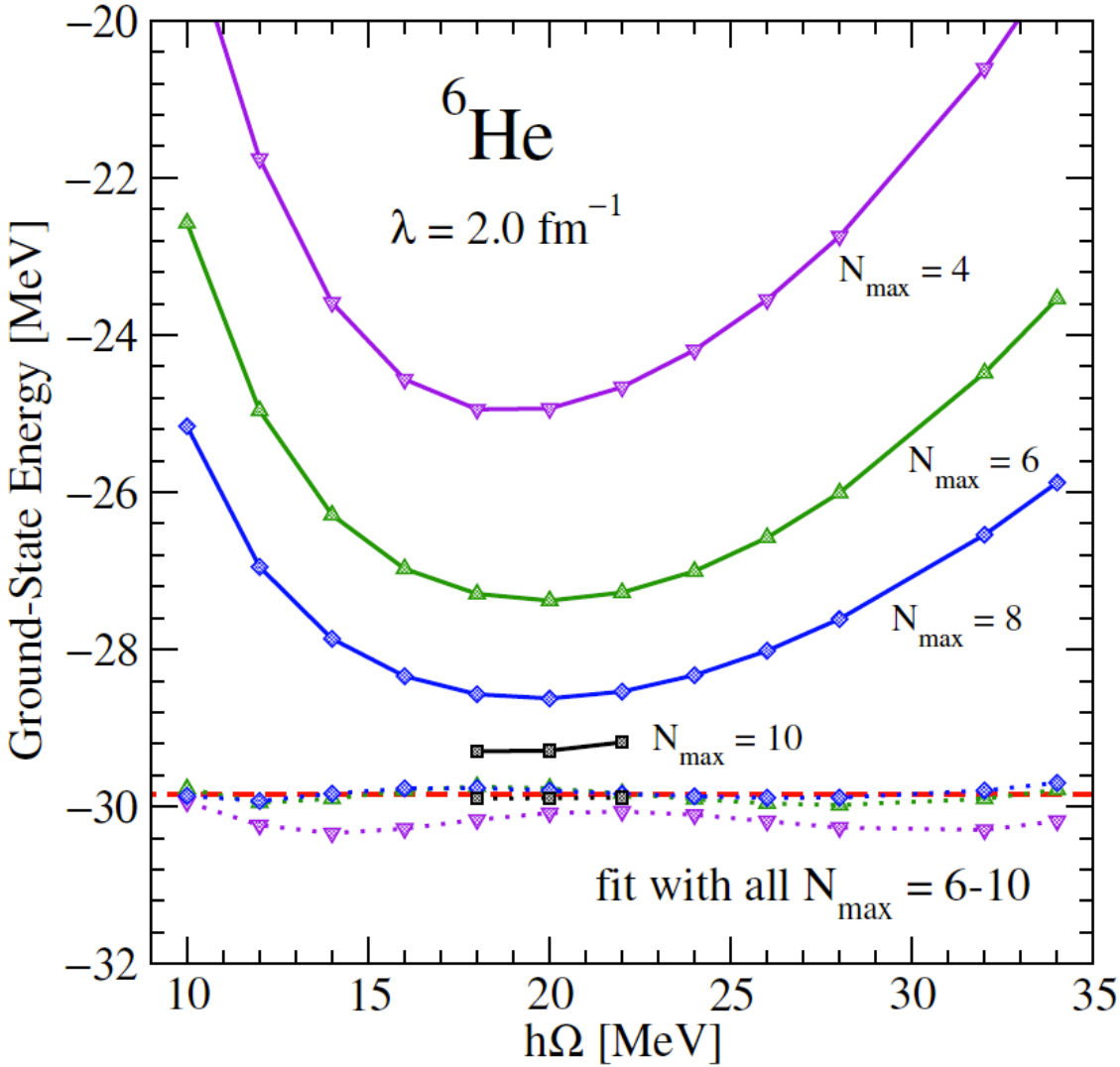
$$E_L = E_\infty + a_0 e^{-2k_\infty L}$$

$$\langle r^2 \rangle_L \approx \langle r^2 \rangle_\infty [1 - (c_0 \beta^3 + c_1 \beta) e^{-\beta}]$$
$$\beta \equiv 2k_\infty L$$





Furnstahl, Hagen, TP, Phys. Rev. C 86, 031301(R) (2012)

Combined UV and IR fit

$$E(\Lambda_{UV}, L) \approx E_\infty + A_0 e^{-2\Lambda_{UV}^2/A_1^2} + A_2 e^{-2k_\infty L}$$