Modeling the Phases of QCD in and beyond mean field theory

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Medium Properties, Chiral Symmetry and Astrophysical Phenomena

Outline

Addressing QCD thermodynamics ($N_f = 2$)

- QCD phase structure
  1. Spontaneous chiral symmetry breaking
  2. Confinement
- Joining the NJL model and the Polyakov loop model

1. Introduction
   - NJL-model
   - Polyakov-loop model
2. Implementing cross-talk between the models
   - NJL-model + Polyakov-loop model ➠ PNJL model
   - An ansatz beyond mean field theory
3. Numeric results
   - Equation of state — Comparison with lattice data
   - The Polyakov loop $\langle \Phi \rangle$ and its complex conjugate $\langle \Phi^* \rangle$
   - The phase diagram (including diquarks)
   - Isovectorial susceptibilities

NJL model

$$\mathcal{L}_{\text{NJL}} = \bar{\psi}\left(\not{p} - m_0\right)\psi - g\left(\bar{\psi}\gamma^\mu\lambda_a\psi\right)\left(\bar{\psi}\gamma_\mu\lambda_a\psi\right)$$

- Free quarks
- Integrated out gluons
- Local colour current interaction
- Chiral symmetry

➠ Local SU(3)$_c$ $\xrightarrow{QCD \to NJL}$ Global SU(3)$_c$ ➠ No confinement in NJL

Spontaneous chiral symmetry breaking

- Hartree-Fock approximation (Fierz-transformation: $H = \frac{3}{4}G$)

$$\mathcal{L} = \bar{\psi}\left[\not{p} - m_0\right]\psi + \frac{G}{2}\left[\bar{\psi}\psi\right]^2 + \frac{H}{2}\left(\bar{\psi}i\gamma_5\tau_2\lambda_2 C\bar{\psi}^T\right)\left(\psi^T C i\gamma_5\tau_2\lambda_2\psi\right)$$

- Bosonization in channels with large 4-quark coupling

$$\Omega_{\text{MF}} = \frac{\sigma^2}{2G} + \frac{|\Delta|^2}{2H} - \frac{T}{2}\sum_{\omega_n}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\,\mathrm{Tr}\log\frac{S^{-1}(\omega_n,\vec{p})}{T}$$

➠ $\sigma = G\,\langle\bar{\psi}\psi\rangle$ ➠ $\Delta = H\,\langle\psi^T C i\gamma_5\tau_2\lambda_2\psi\rangle$

➠ $$S^{-1} = \begin{pmatrix} \not{p} - (m_0 - \sigma) + \gamma_0\mu & \Delta\gamma_5\tau_2\lambda_2 \\ -\Delta^*\gamma_5\tau_2\lambda_2 & \not{p} - (m_0 - \sigma) - \gamma_0\mu \end{pmatrix}$$

Polyakov loop model

- Model for $SU(3)_c$-gauge theory ➠ Confinement
  - ➠ 1st-order ➠ Spontaneous breakdown of $Z(3)$-center of $SU(3)_c$

Order parameter for de-confinement – Polyakov loop

- Polyakov loop $\Phi(\vec{x})$ is a normalized time-like Wilson-line

$$\Phi(\vec{x}) = \frac{1}{N_c}\mathrm{tr}_c L(\vec{x}) \qquad L(\vec{x}) = \mathcal{P}\exp\left\{ i\int_0^\beta \mathrm{d}\tau\, A_4^a(\vec{x})\, t_a \right\}$$

➠ $\langle\Phi\rangle = 0 \iff$ confinement ➠ $\langle\Phi\rangle \neq 0 \iff$ deconfinement

- Define a Ginzburg-Landau effective potential

$$U = U(\Phi,\, \Phi^*, T) \qquad \text{with } \Phi = \frac{1}{N_c}\mathrm{Tr}\exp\left\{ i\frac{A_4^a\, t_a}{T} \right\} \quad \text{and } a \in \{3, 8\}$$

$$\int \mathcal{D}\Phi \int \mathcal{D}\Phi^*\, e^{-U(\Phi,\, \Phi^*, T)} = \int \mathcal{D}A\, e^{-S_{\mathrm{eff}}(\Phi(A), \Phi^*(A), T)}$$

- Effective loop coupling $\propto \Phi^*\Phi$ ➠ $S_{\mathrm{eff}} = -\frac{1}{2} b_2(T)\, \Phi^*\Phi$

Polyakov loop model adjusted to lattice QCD data

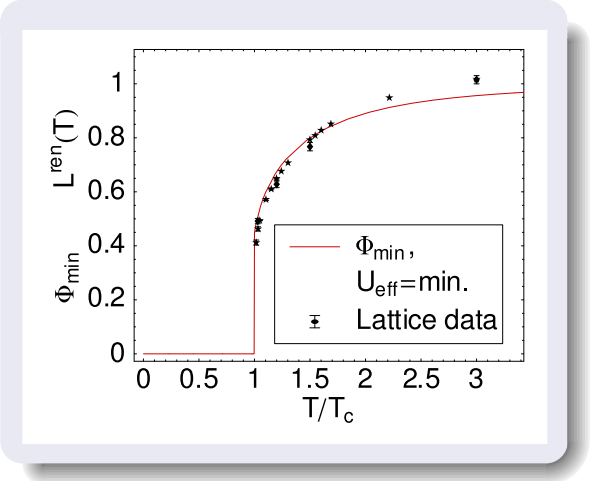
Ansatz for the Polyakov loop potential (K. Fukushima [Fuk04])

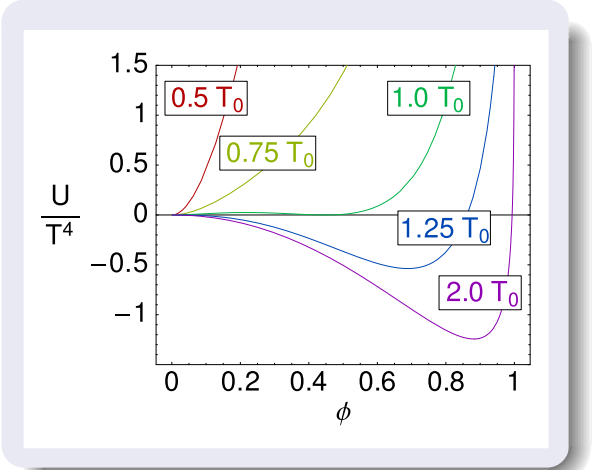
$$\frac{U(\Phi,\Phi^*,T)}{T^4} = -\frac{1}{2}b_2(T)\,\Phi^*\Phi + \frac{1}{4}b_4(T)\,\log\left[J(\Phi,\Phi^*)\right]$$

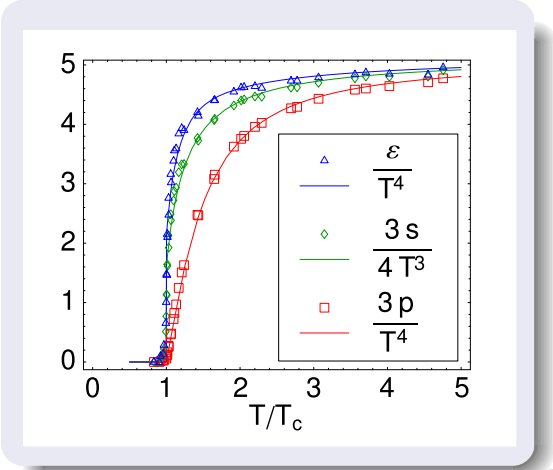
$$J(\Phi,\Phi^*) = 1 - 6\Phi^*\Phi + 4\left(\Phi^{*3} + \Phi^3\right) - 3\left(\Phi^*\Phi\right)^2$$

$$b_4(T) = b_4\left(\frac{T_0}{T}\right)^3 \qquad b_2(T) = a_0 + a_1\left(\frac{T_0}{T}\right) + a_2\left(\frac{T_0}{T}\right)^2$$

- Temperature dependent effective coupling strength $b_2 = b_2(T)$
- Log-term generated by the integration over the $SU(3)_c$ measure
- 1st-order Transition at $T_0 = 270\,\text{MeV}$ in pure gauge lattice QCD







G. Boyd et. al. [B+96], O. Kaczmarek et. al. [KKPZ02, KZ05]

Polyakov loop extended NJL (PNJL)

- Substitute the Matsubara frequencies ω_n by $\omega_n + A_4$
  - Formal substitution $\mu \to \mu - iA_4$ *after* Matsubara summation
  - Analogy to the QCD gauge coupling

$$\Omega_{\mathrm{MF}} = \Omega_{\mathrm{NJL}}\big|_{\mu \to \mu - iA_4} + U(\Phi, \Phi^*, T)$$

From NJL to PNJL

- In SU(3)$_c$-gauge theory: $T_c = 270\,\mathrm{MeV}$
- In NJL: $T_c \approx 177\,\mathrm{MeV}$, $T_{\mathrm{CEP}} \approx 38\,\mathrm{MeV}$
- In PNJL: $T_c \approx 215\,\mathrm{MeV}$, $T_{\mathrm{CEP}} \approx 135\,\mathrm{MeV}$
- Lattice: $T_c \approx 202\,\mathrm{MeV}$ O. Kaczmarek et. al. [KZ05]

"Confinement" in the PNJL model:

- Thermodynamic suppression of free quarks by the Polyakov loop
  - At $\langle \Phi \rangle = 0$ ($\Longleftrightarrow$ "confinement") PNJL models a gas with particle mass $m = 3\,M \approx M_{\mathrm{N}}$

Constraints imposed by MF approach

The fermion sign problem

- Minimal substitution: $\omega_n \to \omega_n + A_4$
- ➠ Inverse propagator $S^{-1}(\mu \to \mu - iA_4)$ is *not* hermitian
- ➠ The fermion determinant is a complex quantity
- ➠ Weights are *not* positive definite (➠ "sign problem")

- Integrating out all fields:
- ➠ Partition function $Z \in \mathbb{R}$ despite complex weights

MF: One field configuration carries all weight

➠ Subtle cancellation of imaginary parts in Z is disturbed

? Interpretation of Z and thermodynamic potential Ω

Idea: Do not discard the integral over bosonic field configurations

➠ Conserve the cancellation of imaginary parts in Z and Ω

Gaussian approximation

A: Truncate the bosonized action $S_{\text{bos.}}$ at 2nd order in the fields

$$S_{\text{bos.}} \approx S_{\text{trunc.}} = \frac{V}{T}\left(\omega_0 + \omega_1 \cdot \xi + \frac{1}{2}\xi \cdot \omega_2 \cdot \xi\right)$$

B: Treat higher orders as *perturbations* $S_{\text{I}} = S_{\text{bos.}} - S_{\text{trunc.}}$

C: Choose the field configuration to expand about

- Maximal convergence of the perturbative series
- ➠ Expand about the configuration where $\left|e^{-S_{\text{trunc.}}}\right| = \max.$
- ➠ $\left.\frac{\partial \operatorname{Re}\Omega_{\text{MF}}}{\partial \phi}\right|_{\phi \in \{\sigma, \Delta, \phi_3, \phi_8\}} = 0$ ➠ Self consistency equations

- Expansion about the minimum after SSB ➠ *Corrections to MF*

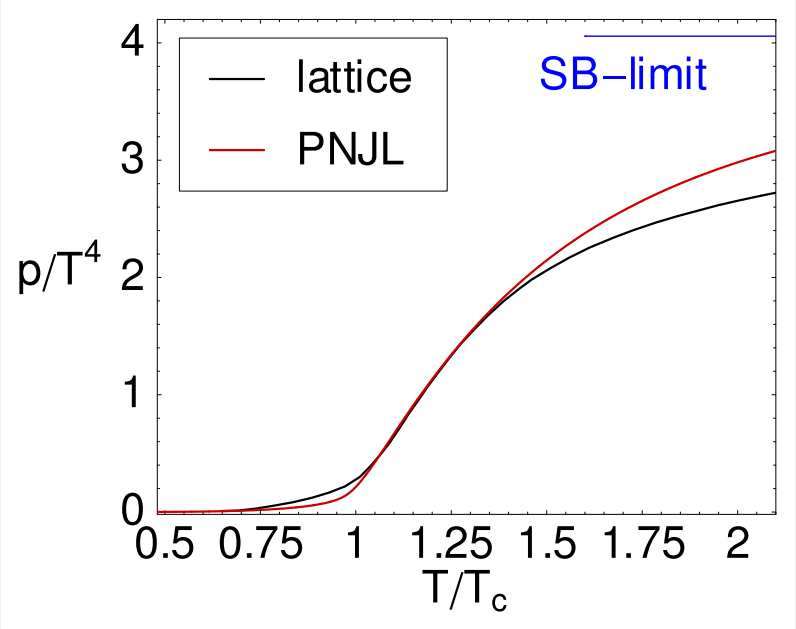
Diagrams are systematically ordered...

1. ...by the *thermodynamic* expansion $\propto (T/V)^{\alpha}$
2. ...by the number of *source term* insertions δ^{β},
   where δ is defined as $\delta = -i\,[\omega_2]^{-1} \cdot \omega_1$

The pressure and its moments

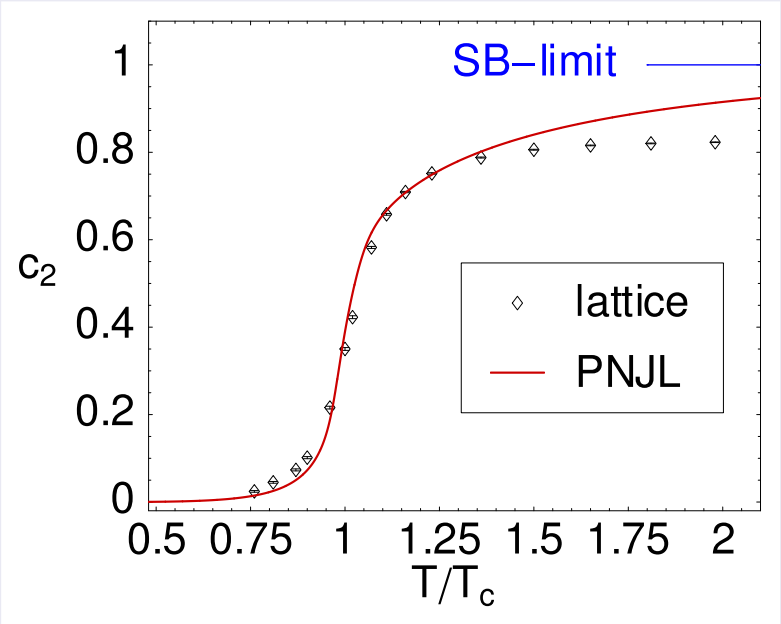
Pressure and c_2, S. R., C. Ratti, W. Weise [RRW07]

The pressure p/T^4



- Lattice data:
  F. Karsch et. al. [KLP00]

The second moment c_2



- Lattice data:
  C. R. Allton et. al. [A$^+$05]

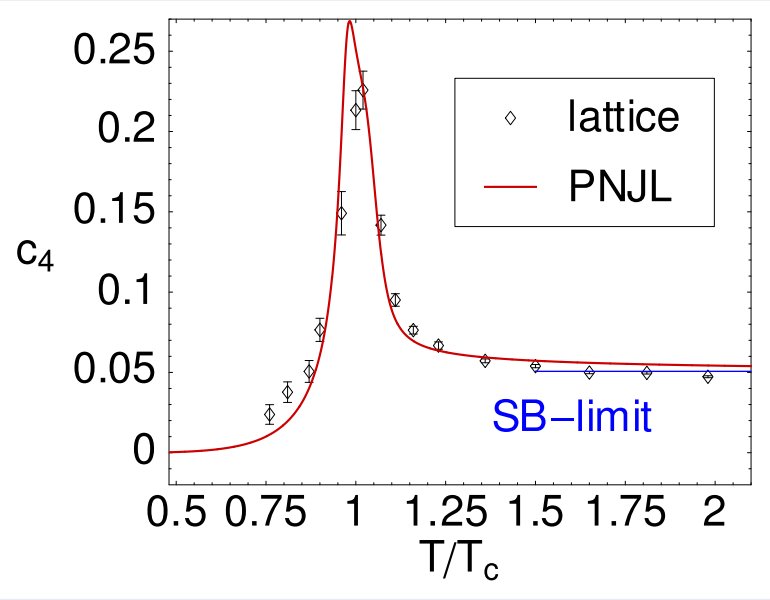
$$\Delta\left(\frac{p}{T^4}(\mu)\right) = \frac{p}{T^4}\Big|_{T,\mu} - \frac{p}{T^4}\Big|_{T,0} = \sum_{p=1}^{\infty} c_p(T)\left(\frac{\mu}{T}\right)^p,$$

$$\Rightarrow c_2 = \frac{1}{2!}\frac{1}{T^2}\frac{\partial^2 p}{\partial \mu^2}$$

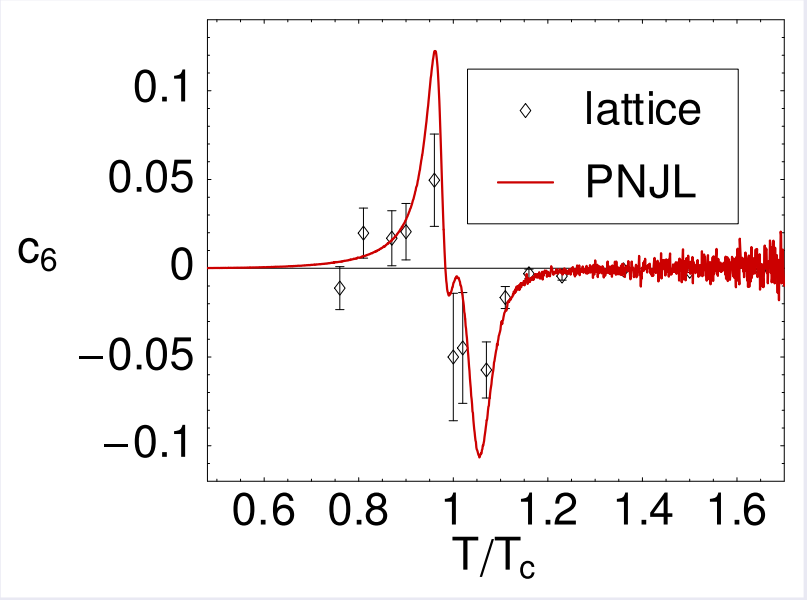
The pressure difference and its moments

c_4 and c_6, S. R., C. Ratti, W. Weise [RRW07]

The fourth moment c_4



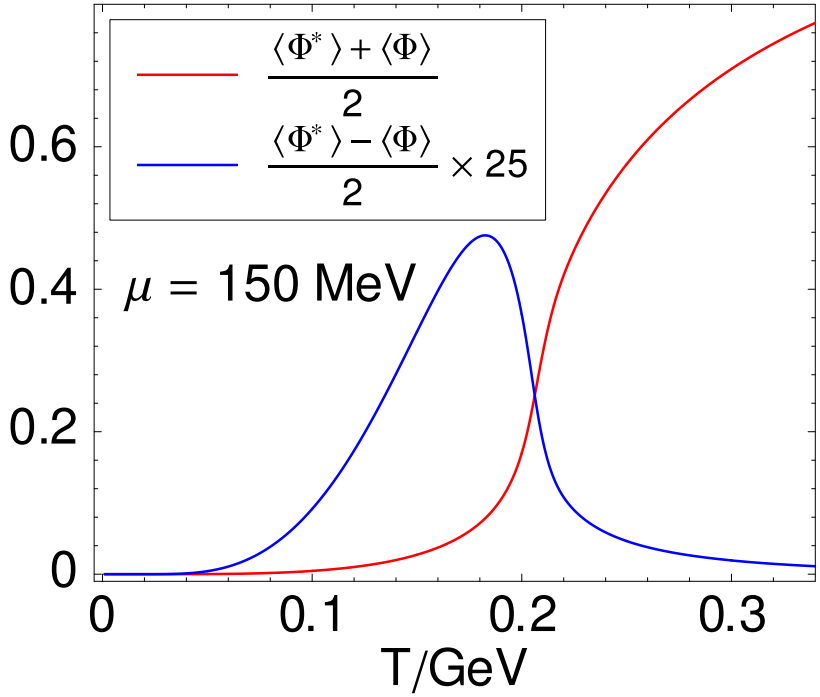
The sixth moment c_6

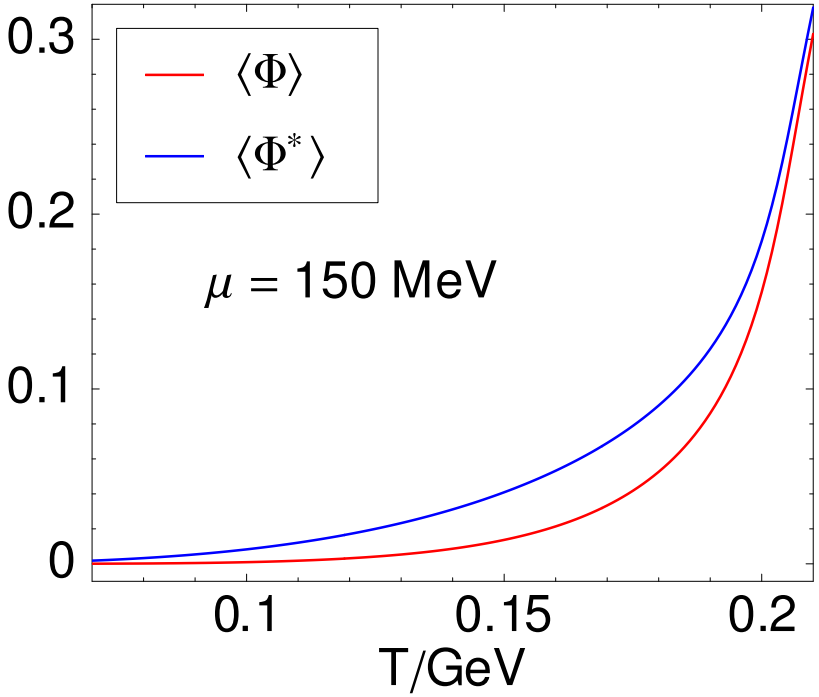


- Lattice data from the Bielefeld-Swansea coll. [A$^+$05]

$$\Rightarrow c_4 = \frac{1}{4!}\frac{\partial^4 p}{\partial \mu^4} \qquad \Rightarrow c_6 = \frac{1}{6!}T^2\frac{\partial^6 p}{\partial \mu^6}$$

Expectation values of the Polyakov loop $\langle\Phi\rangle$ and $\langle\Phi^*\rangle$



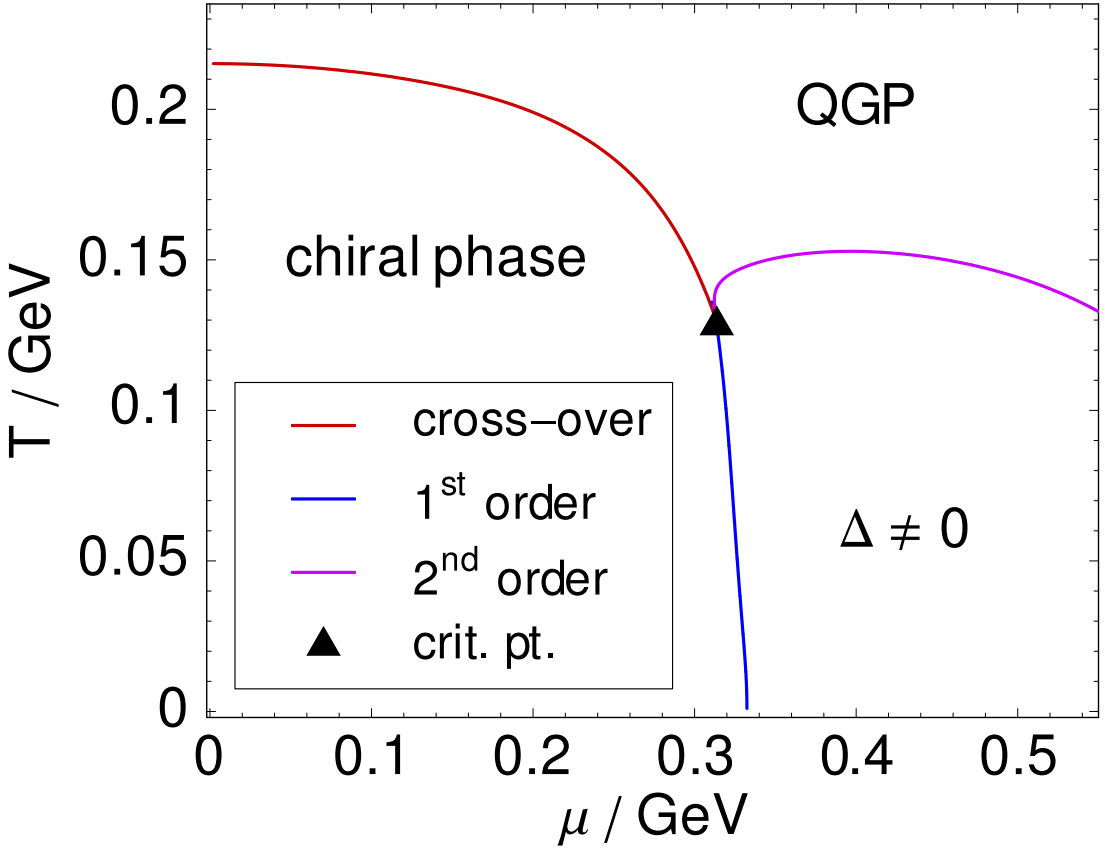


In mean field	MF + corrections
• $\langle\Phi\rangle_{\text{MF}} = \langle\Phi^*\rangle_{\text{MF}}$ • No split of $\langle\Phi\rangle$ and $\langle\Phi^*\rangle$	• $\langle\Phi\rangle \in \mathbb{R}$ and $\langle\Phi^*\rangle \in \mathbb{R}$ • $\langle\Phi\rangle \neq \langle\Phi^*\rangle$ at $\mu \neq 0$

➠ Fluctuation effects beyond mean field produce $\langle\Phi\rangle \neq \langle\Phi^*\rangle$

The phase diagram

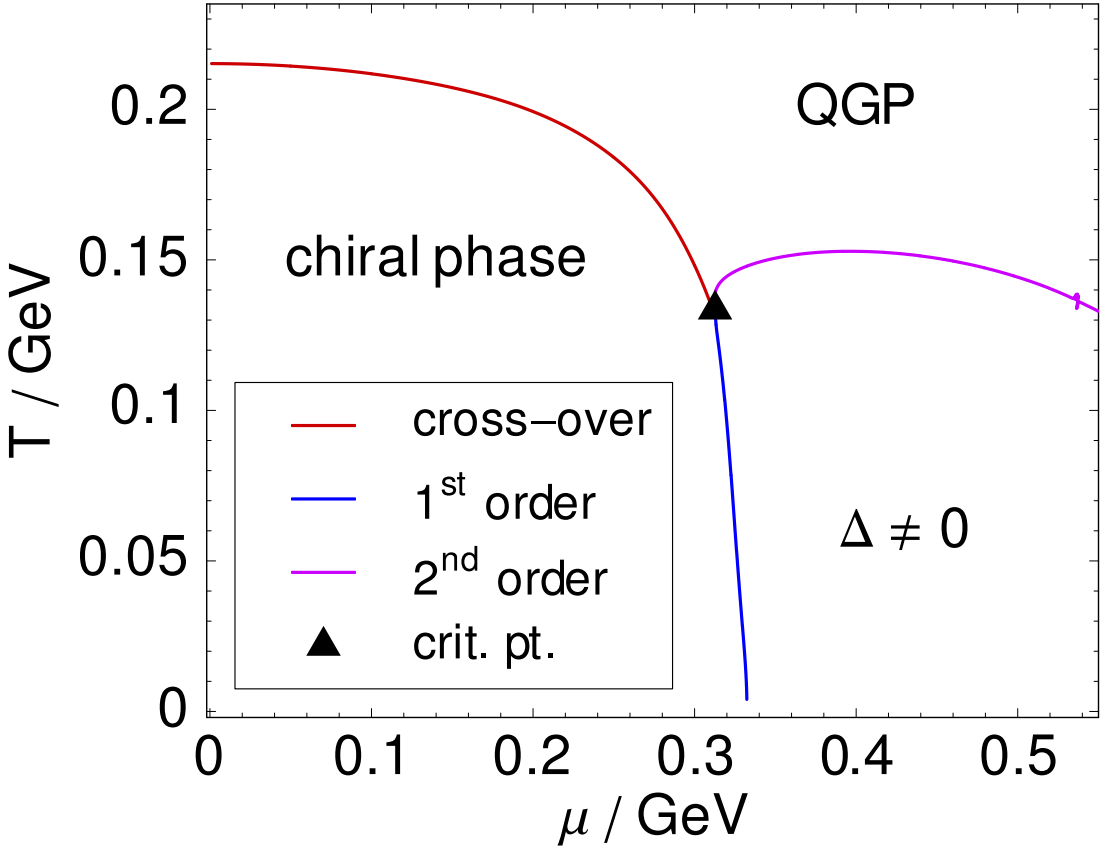
S. R., C. Ratti, W. Weise [RRW07]



- No back reaction of fluctuations on SSB included

The phase diagram (in mean field)

S. R., C. Ratti, W. Weise [RRW07]



- The phase diagram is almost unaffected by the corrections
- ➠ Phase structure is governed by *non*-perturbative SSB

Introduction of a non-vanishing pion condensate [ZL06]

Trading the diquark condensate for a pion condensate

$$\mathcal{L}_{\text{NJL}} = \bar{\psi}\left(\not{p} + \gamma_0\hat{\mu} - \hat{m}_0 - i\hat{\lambda}\gamma_5\tau_1\right)\psi + \frac{G}{2}\left[\left(\bar{\psi}\psi\right)^2 + \left(\bar{\psi}i\gamma_5\vec{\tau}\psi\right)^2\right]$$

$$\hat{\mu} = \mu\,\mathbb{1} + \mu^{\mathrm{I}}\sigma_3 \qquad \hat{m}_0 = m_0\,\mathbb{1} \qquad \hat{\lambda}_0 = \lambda\,\mathbb{1}$$

- no diquark coupling term ➠ pseudoscalar, isovector channel

$$\Rightarrow \mathcal{H}_{\text{PNJL}} = -i\psi^\dagger\left(\vec{\alpha}\cdot\vec{\nabla} + \gamma_4 m_0 - \phi\right)\psi + \frac{G}{2}\left[\left(\bar{\psi}\psi\right)^2 + \left(\bar{\psi}i\gamma_5\vec{\tau}\psi\right)^2\right] + \mathcal{U}\left(\Phi, \Phi^*, T\right)$$

➠ The thermodynamic potential (Zhang and Liu [ZL06]):

$$\Omega = \mathcal{U}\left(\Phi, \Phi^*, T\right) + \frac{\sigma^2 + \pi^2}{2G} - T\sum_n \int \frac{\mathrm{d}^3p}{(2\pi)^3}\,\mathrm{Tr}\log\frac{S^{-1}\left(i\omega_n, \vec{p}\right)}{T}$$

$$S^{-1} = \begin{pmatrix} (i\omega_n + \tilde{\mu} + \mu^{\mathrm{I}})\gamma_0 - \vec{\gamma}\cdot\vec{p} - M & -i\gamma_5 N \\ -i\gamma_5 N & (i\omega_n + \tilde{\mu} - \mu^{\mathrm{I}})\gamma_0 - \vec{\gamma}\cdot\vec{p} - M \end{pmatrix}$$

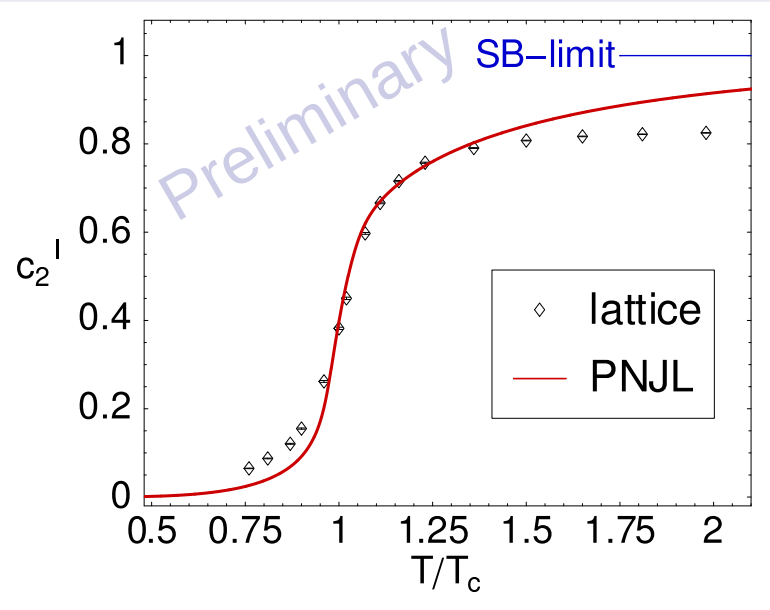
$$\tilde{\mu} = \mu - iA_4 \qquad M = m_0 - 2G\,\sigma \qquad N = \lambda - 2G\,\pi$$

$$\sigma \simeq \langle\bar{\psi}\psi\rangle \qquad \pi \simeq \langle\bar{\psi}i\gamma_5\tau_1\psi\rangle$$

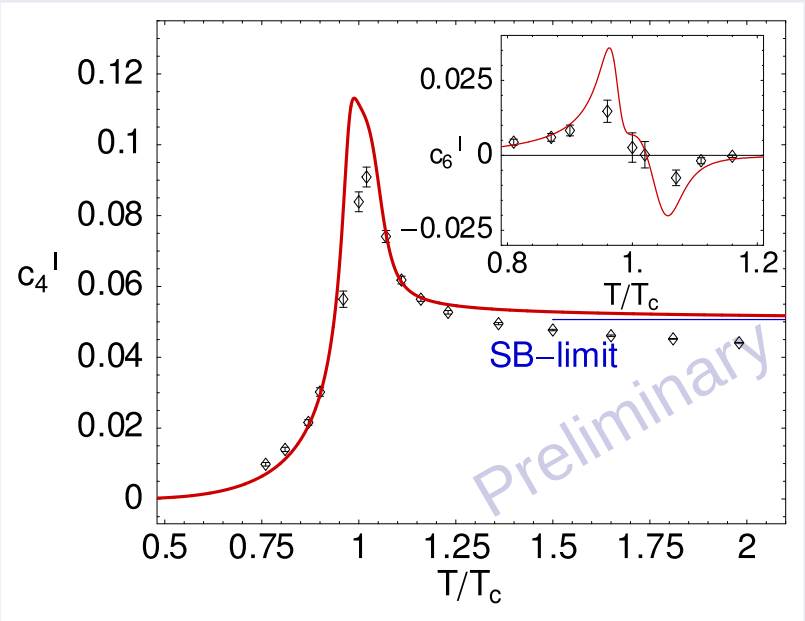
The expansion of the pressure difference in isospin

c_2^{I}, c_4^{I} and c_6^{I} — No isospin breaking: $\lambda = 0$

The coefficient c_2^{I}



The coefficient c_4^{I} and c_6^{I}



- Lowest order corrections to MF included
  - ✔ Fluctuations of the $q = 0$-mode included to lowest order
  - ✘ No fluctutuations from mesonic $q \neq 0$-modes, yet . . .
- Lattice data from the Bielefeld-Swansea coll. [A+05]

$$c_2^{\mathrm{I}} = \frac{1}{2!}\frac{1}{T^2}\frac{\partial^2 p}{\partial \mu_{\mathrm{I}}^2} \qquad c_4^{\mathrm{I}} = \frac{1}{4!}\frac{\partial^4 p}{\partial \mu^2 \partial \mu_{\mathrm{I}}^2} \qquad c_6^{\mathrm{I}} = \frac{1}{6!}T^2\frac{\partial^6 p}{\partial \mu^4 \partial \mu_{\mathrm{I}}^2}$$

Conclusion

- PNJL:
  - ✔ Chiral symmetry breaking
  - ✔ Confinement
- Corrections to mean field results
  - ✔ Consistently fixes the fermion sign problem order by order
  - ✘ Base for further investigation on fluctuations
- Numeric Results
  - ✔ Astonishing agreement with QCD lattice data
  - ✔ $\langle\Phi\rangle$ and $\langle\Phi^*\rangle$ are non-degenerate at $\mu \neq 0$
  - ✔ Corrections to the phase diagram beyond MF are small
  - ✔ Isovectorial susceptibilies from PNJL and lattice
- Outlook
  - ✘ Effect of pionic fluctuations ($q \neq 0$) on isovector susceptibilities
  - ✘ Improving the regularization scheme (Thomas Hell)
  - ✘ $2 + 1$ flavors

The end.

Thank you for your attention

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Parameter fixing

NJL parameters

- $G = 10.08\,\text{MeV}$
- $\Lambda = 651\,\text{MeV}$
- $m_0 = 5.5\,\text{MeV}$

$\Longleftrightarrow$

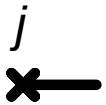
- $m_\pi = 140.5\,\text{MeV}$
- $f_\pi = 94.0\,\text{MeV}$
- $\left|\langle\bar{\psi}\psi\rangle\right|^{1/3} = 251\,\text{MeV}$

Polyakov loop model parameters

$a_0 = 3.51$	$a_1 = -2.47$	$a_2 = 15.2$	$b_4 = -1.75$
SB-limit	$\Delta a_1 \approx 6\%$	$\Delta a_2 \approx 3\%$	$\Delta b_4 \approx 2\%$

Establishing Feynman rules

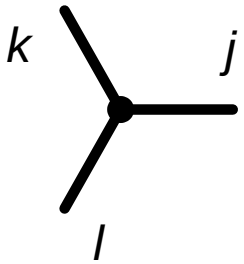
2. Separate free and perturbative parts:
   - $S_0 = \frac{V}{T}\left(\omega_0 + \vec{\omega}_1^T \vec{\xi} + \frac{1}{2}\vec{\xi}^T \overleftrightarrow{\omega}_2 \vec{\xi}\right)$
   - Note: $\mathrm{Re}\,\omega_1 = 0$, $\mathrm{Im}\,\omega_1 \neq 0$ for the P-loop parameters $\phi_{3,8}$
   - $S_\mathrm{I} = \frac{V}{T}\sum_{k=3}^{\infty} \frac{1}{k!}\omega_k\, \vec{\xi}^k$
3. Introduction to the diagrammatic
   - Graphs produced by the "free" part



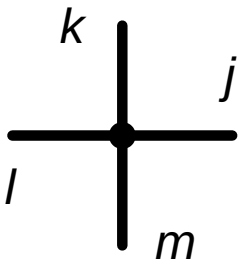
$$= \frac{\partial S}{\partial \xi_j}$$

$$= \left[\frac{\partial^2 S}{\partial \xi_j \partial \xi_k}\right]^{-1}$$

   - "Interaction" graphs



$$= -\frac{\partial^3 S}{\partial \xi_j \partial \xi_k \partial \xi_l}$$



$$= -\frac{\partial^4 S}{\partial \xi_j \partial \xi_k \partial \xi_l \partial \xi_m}$$

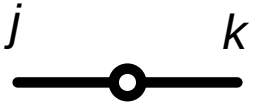
$\vdots$ $\qquad$ $\vdots$

Additional Feynman rules

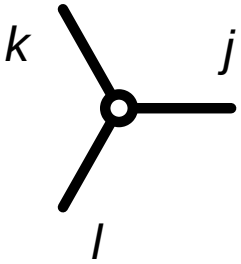
Evaluation of expectation values of arbitrary functions

3. Introduction to the diagrammatic (cont'd)
   - Graphs needed to evaluate $\langle f(\xi)\rangle$ ➠ Include exactly one circle

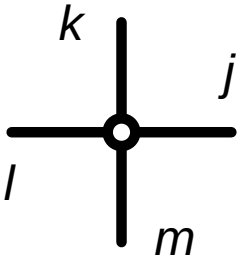
$$= \frac{\partial f}{\partial \xi_j}$$



$$= \frac{\partial^2 f}{\partial \xi_j \partial \xi_k}$$



$$= \frac{\partial^3 f}{\partial \xi_j \partial \xi_k \partial \xi_l}$$



$$= \frac{\partial^4 f}{\partial \xi_j \partial \xi_k \partial \xi_l \partial \xi_m}$$

$$\vdots \qquad\qquad \vdots$$

   - Evaluation of $\langle g(\xi)^2\rangle - \langle g(\xi)\rangle^2$ ➠ Include exactly two circles

➠ Split graphs above into two parts, i. e. $f \longrightarrow g^2$

$$g^2(\theta)\Big|_{\theta=\theta_{\mathrm{MF}}} = g\,g$$

$$\partial_\theta\, g^2(\theta)\Big|_{\theta=\theta_{\mathrm{MF}}} = 2g\,g^{(1)}$$

$$\partial_\theta^2\, g^2(\theta)\Big|_{\theta=\theta_{\mathrm{MF}}} = 2\,g\,g^{(2)} + 2\,g^{(1)}\,g^{(1)}$$

Formal ordering of the perturbative terms

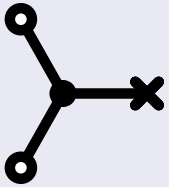
1. Thermodynamic expansion: α counts powers in $\frac{T}{V}$
   - $\alpha(S) = -1$ and $\alpha(\Omega) = 0$
2. Source terms: β counts powers in $\sum_k \left[\frac{\partial^2 S}{\partial\xi_j \partial\xi_k}\right]^{-1} \frac{\partial S}{\partial\xi_k}$

Orders evaluated in the numeric calculations: $\alpha = 0$ and $\beta = 0, 1$

- Fractions give multiplicity factors

$$\Omega = \Omega_{\mathrm{MF}} - \frac{1}{2}\frac{T}{V}$$ $$= \Omega_{\mathrm{MF}} - \frac{1}{2}\left(\frac{\partial\Omega_{\mathrm{MF}}}{\partial\theta}\right)^T \left[\frac{\partial^2\Omega_{\mathrm{MF}}}{\partial\theta^2}\right]^{-1} \frac{\partial\Omega_{\mathrm{MF}}}{\partial\theta}\bigg|_{\theta=\theta_{\mathrm{MF}}}$$

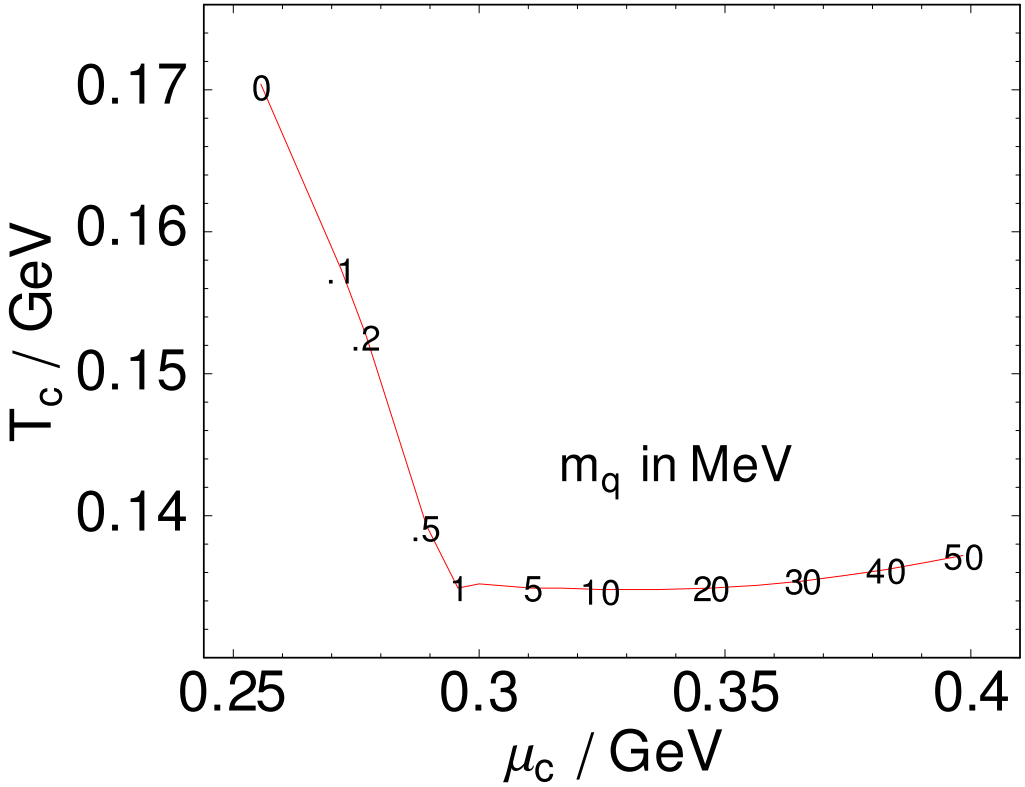
$$\langle f\rangle = f(\theta_{\mathrm{MF}}) +$$ $$= f(\theta_{\mathrm{MF}}) + \left(\frac{\partial\Omega_{\mathrm{MF}}}{\partial\theta}\right)^T \left[\frac{\partial^2\Omega_{\mathrm{MF}}}{\partial\theta^2}\right]^{-1} \frac{\partial f}{\partial\theta}\bigg|_{\theta=\theta_{\mathrm{MF}}}$$

$$\langle \Delta g^2\rangle = \frac{1}{2}\times 2$$ $$+ \frac{1}{2}\times 4$$ $$+ \frac{1}{2}\times 2$$ 

- Lowest order in α completely cancels ➠ Susceptibilities $\propto \frac{T}{V}$
- Additional factors from differentiation of $g^2(\xi)$ (cutting one vertex into two parts)

The critical endpoint and its quark mass dependence

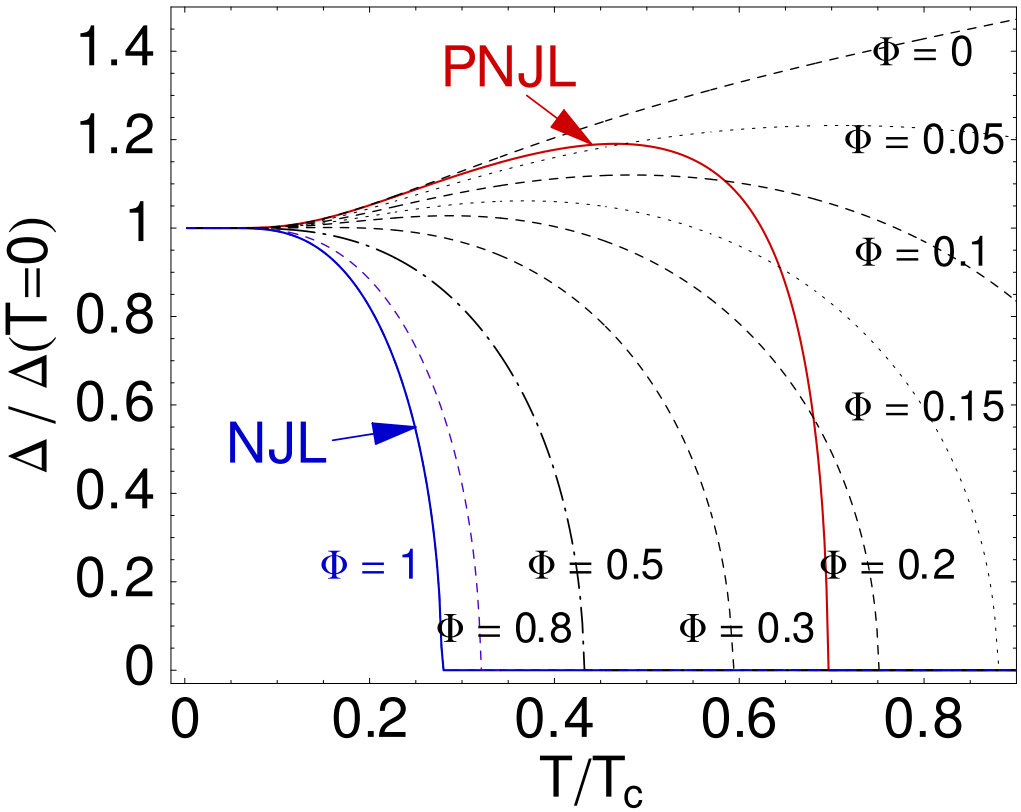
An analysis in mean field approximation



- Diquark phase stabilizes the critical point at high temperatures
  - ➠ T_c is stabilized at $m_0 \gtrsim 1\,\mathrm{MeV}$

Influence of the loop $\langle\Phi\rangle$ on the diquark gap Δ

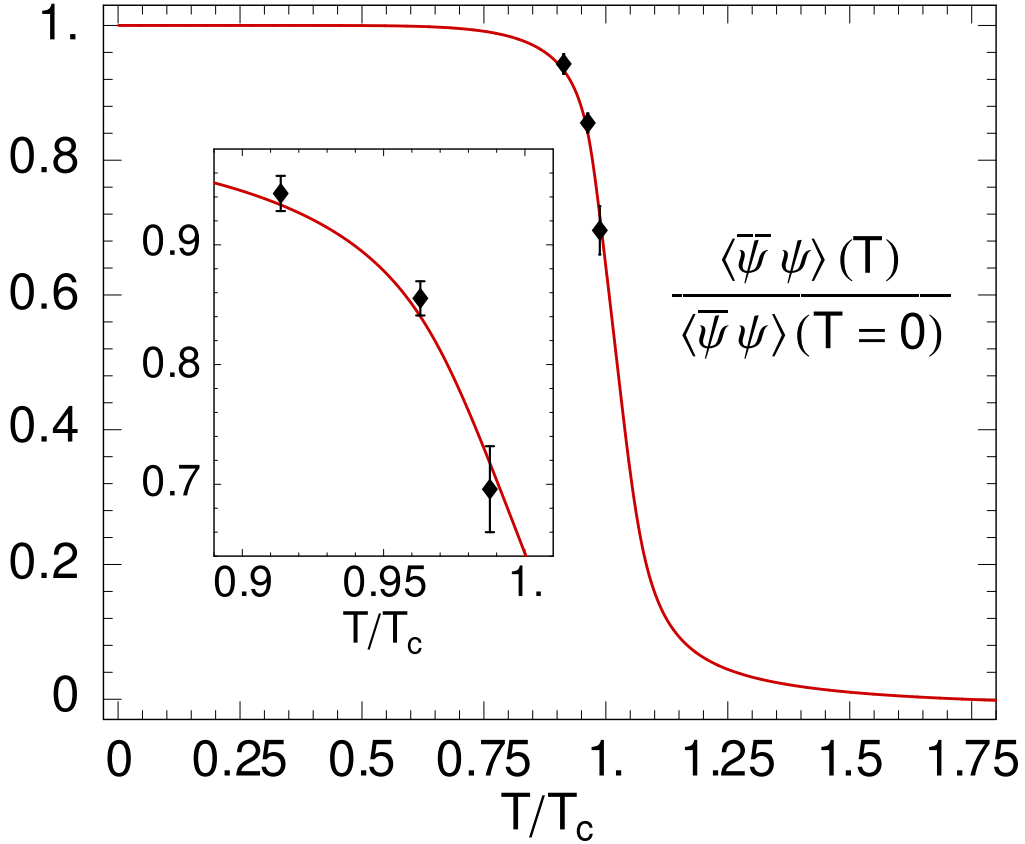
An analysis in mean field approximation (S. R., C. Ratti, W. Weise [RRW07])



- The diquark gap is enlarged for $\langle\Phi\rangle$ small (confinement)
- $\langle\Phi\rangle = 1$ (deconfinement) coincides with NJL case
- Sensitive interplay of pairing quarks (at the Fermi surface) and the thermodynamic suppression of quarks by the loop $\langle\Phi\rangle$ in the center of the Fermi sphere ($E - \mu \lessapprox T$)

The temperature dependence of the chiral condensate

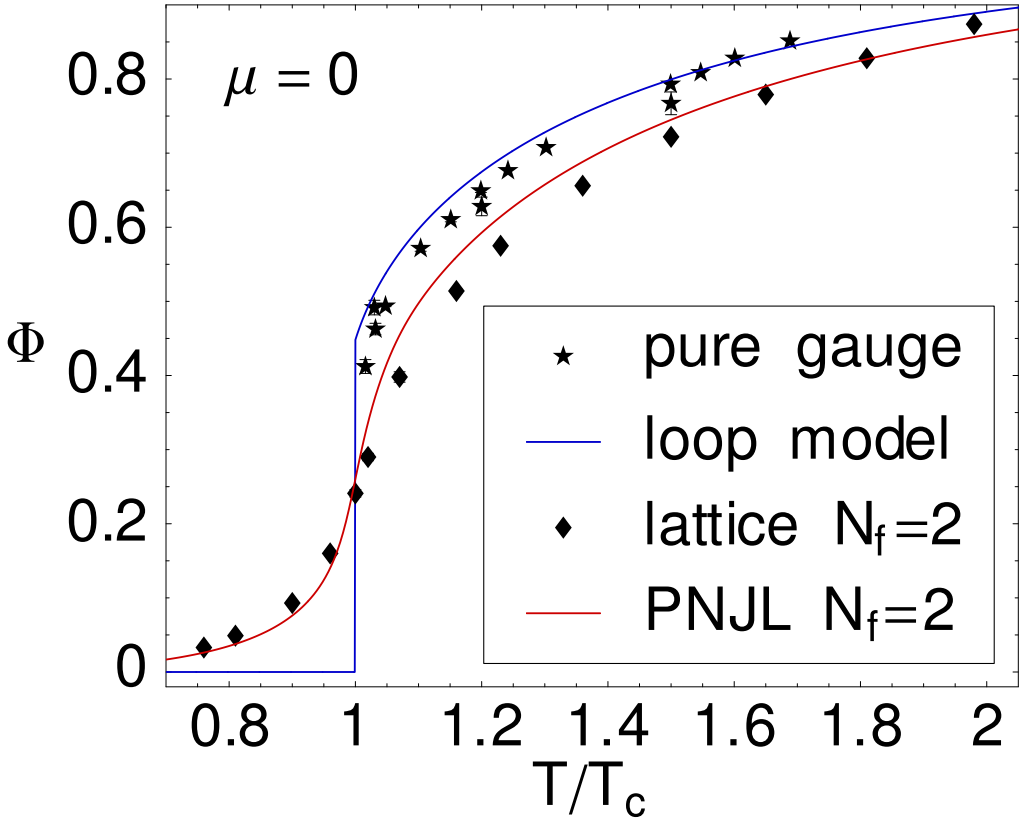
A comparison with lattice data in $n_f = 2$



- Lattice data in $n_f = 2$: Boyd et. al. [B$^+$95]
- Solid red line: PNJL-model calculation (S. R., C. Ratti, W. Weise [RRW07])

The temperature dependence of the Polyakov loop

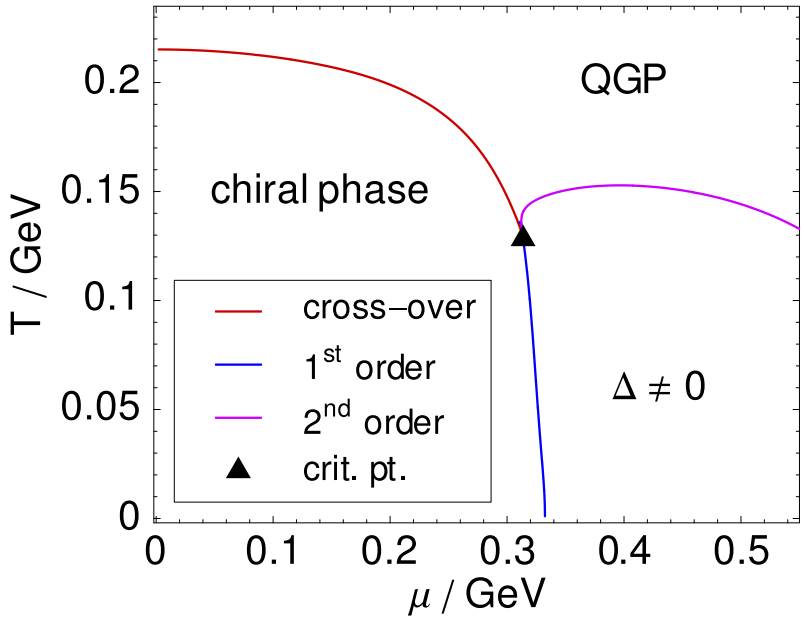
Pure gauge theory in comparison with $n_f = 2$



- Lattice data: O. Kaczmarek et. al. [KZ05]

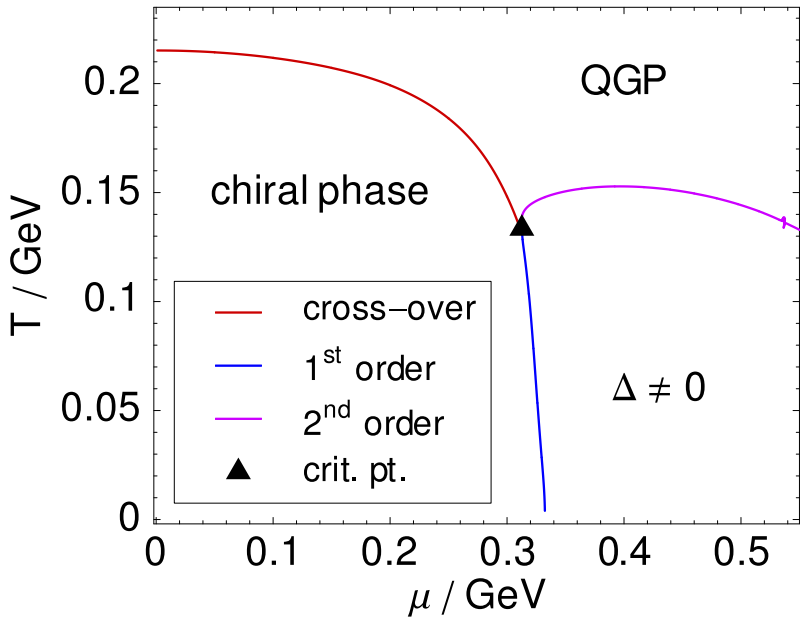
The phase diagram

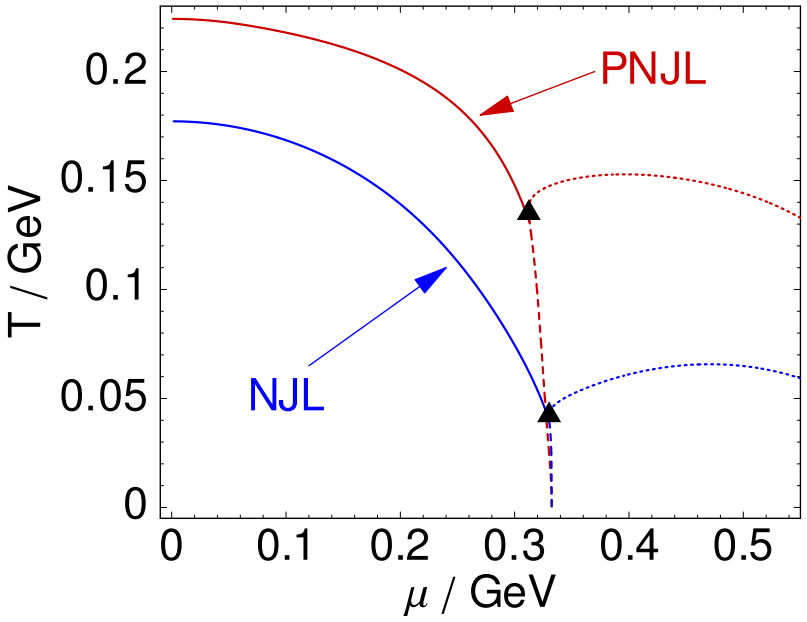
S. R., C. Ratti, W. Weise [RRW07]

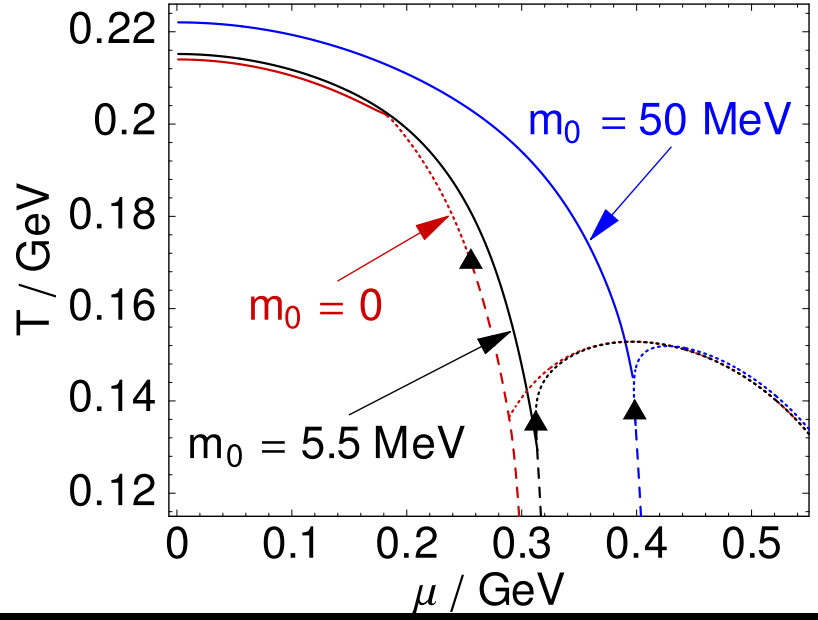


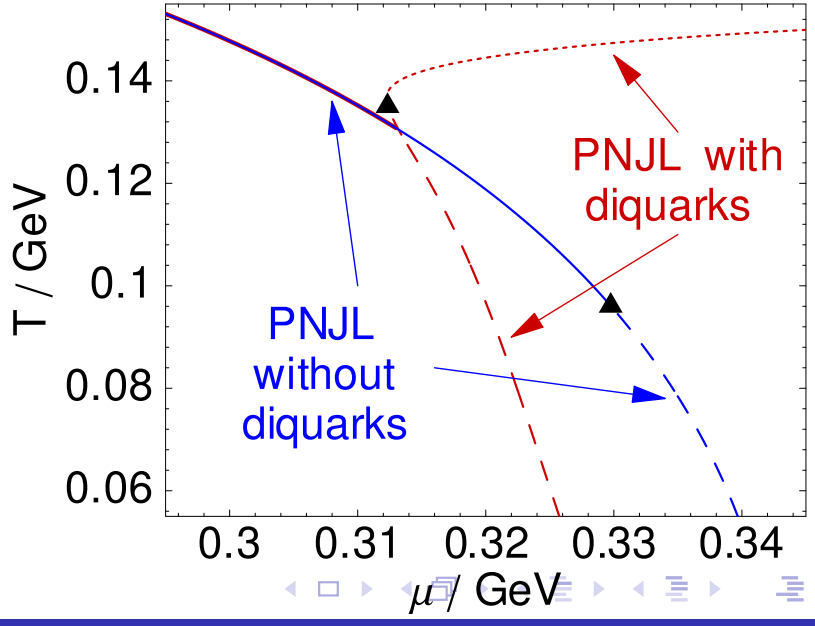
The phase diagram (in mean field)

S. R., C. Ratti, W. Weise [RRW07]





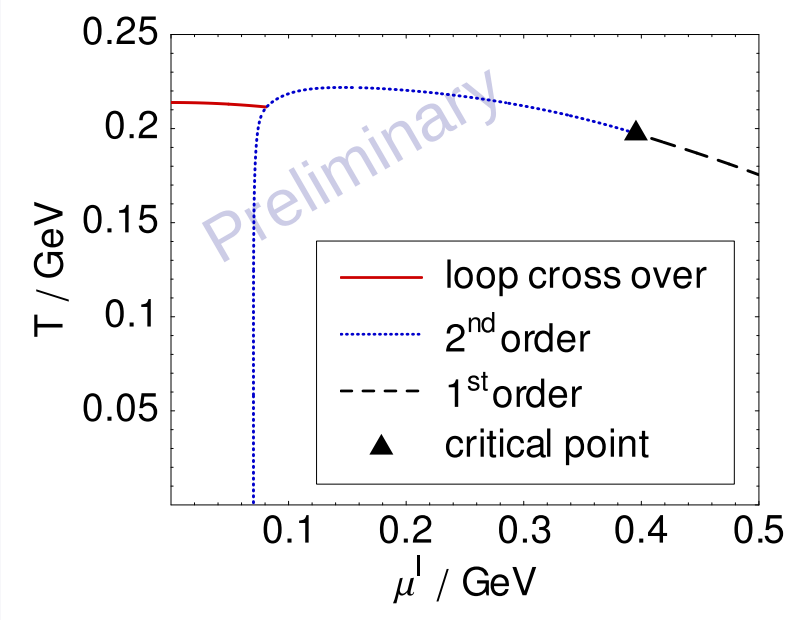




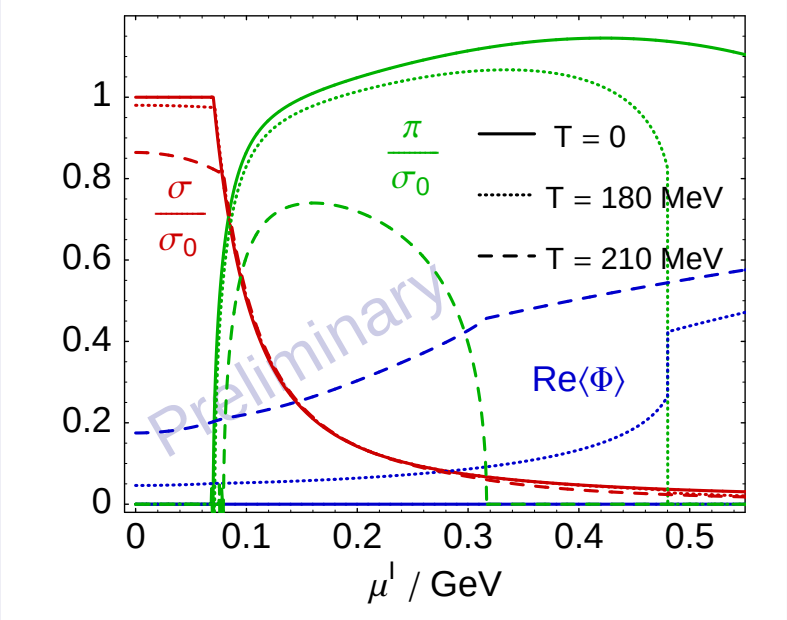
Non-vanishing isovector chemical potential

➠ No explicit isospin symmetry breaking term: $\lambda = 0$ (see also Zhang and Liu [ZL06])

Phase diagram in the (T, μ^{I})-plane



The onset of the pion condensation



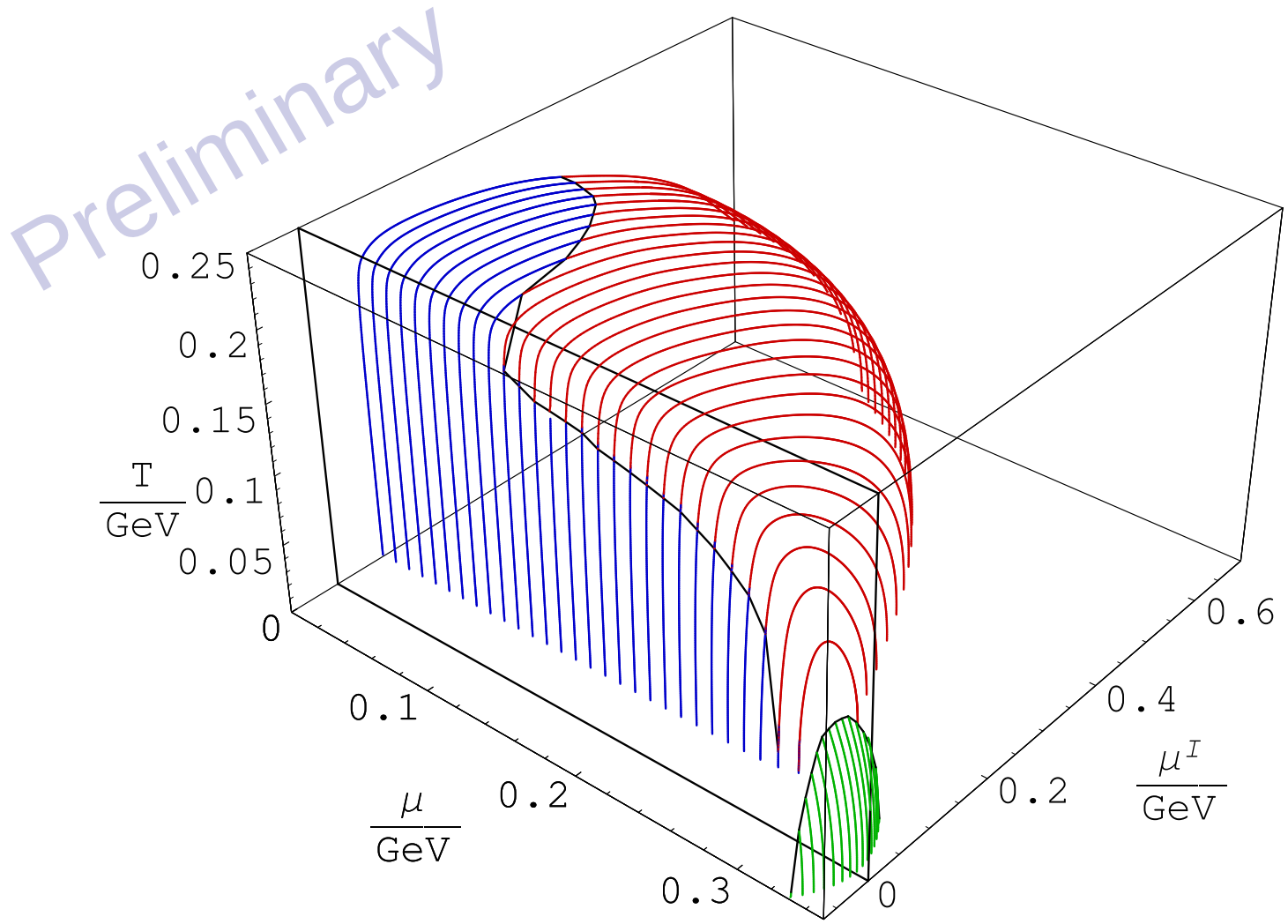
- At low μ^{I} ➠ 2nd order phase transition
- At large μ^{I} ➠ 1st order transition
- Critical point: $\mu^{\mathrm{I}} = 0.395\,\mathrm{GeV}$, $T = 0.197\,\mathrm{GeV}$, $(\mu = 0)$

! Considering the phases in $(T, \mu, \mu^{\mathrm{I}})$ with Δ, $\pi \neq 0$ is difficult

? Is there a tri-critical point, where Δ- and π-phases meet?

A three dimensional impression of pion condensation

➠ No explicit isospin symmetry breaking term: $\lambda = 0$



- 1st-order lines in red and green
- 2nd-order transition in blue

➠ *No connection* of the 1st-order transition at large μ and μ^I