What were the crucial steps?

in DIS Limit $\left(\begin{array}{ll} Q^2 \to \infty & x = \dfrac{Q^2}{2M\nu} = \text{fixed} \\ \nu \to \infty & \end{array} \right)$

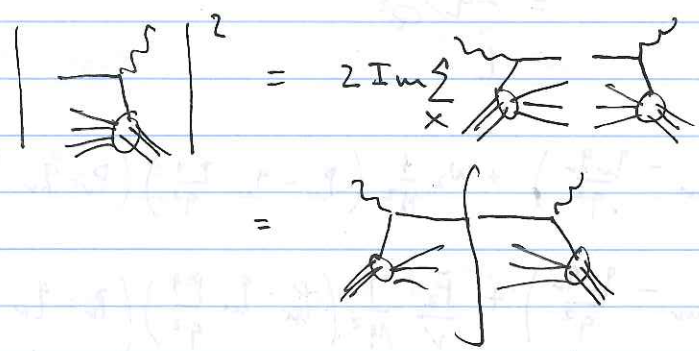
the fundamental constituents of the proton must be weakly interacting spin-½ fermions

What also makes this useful is the PDFs are universal functions of a hadron (independent of process

$\boxed{\text{Tues, 25 March}}$

<u>Recap</u>

How did we arrive at form for $W_{\mu\nu}$?



$$W_{\mu\nu} = \frac{1}{2M} \frac{1}{2} \sum_\sigma \sum\!\!\!\!\!\!\int_\Gamma \langle p,\sigma_{out} | J_\mu | \Gamma \times \Gamma | J_\nu | p,\sigma_{in} \rangle$$

normalization

$\langle p | p' \rangle = 2E_p (2\pi)^3 \delta^3(\vec{p} - \vec{p}')$

$p_{in} = p_{out}$ = forward scattering

$\frac{1}{2}\sum_\sigma$ = unpolarized

$|\Gamma\rangle$ = intermediate hadronic state, Γ (or X)

$\sum\!\!\!\!\!\!\int_X$ sum over all possible intermediate states $\equiv$ inclusive

Lets look at the ratio

$$R_{L/T} = \frac{\sigma_L}{\sigma_T}$$

$$\sigma_L = (\varepsilon_L^\mu)^* W_{\mu\nu} \varepsilon_L^\nu$$

Lab frame: $\varepsilon_L^\mu = \frac{1}{Q}(q_3, 0, 0, q_0)$, $P_\mu = (M, 0, 0, 0)$

$(Q^2 = -q^2)$ $\qquad q_\mu = (q_0, 0, 0, q_3)$

$$\varepsilon_L \cdot q = 0$$

$$\varepsilon_L \cdot p = \frac{q_3 M}{Q}$$

$$\varepsilon_L^{*\mu} g_{\mu\nu} \varepsilon_L^\nu = \frac{1}{Q^2}(q_3^2 - q_0^2)$$

$$= \frac{1}{Q^2}(q_3^2 - \nu^2)$$

$$= -q^2/Q^2$$

$$= \underline{1}$$

$$W_{\mu\nu} = -W_1\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + W_2 \frac{1}{M^2}\left(P_\mu - q_\mu \frac{P\cdot q}{q^2}\right)\left(P_\nu - q_\nu \frac{P\cdot q}{q^2}\right)$$

$$= -\frac{F_1}{M}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + \frac{F_2}{\nu}\frac{1}{M^2}\left(P_\mu - q_\mu \frac{P\cdot q}{q^2}\right)\left(P_\nu - q_\nu \frac{P\cdot q}{q^2}\right)$$

$$W_L \equiv (\varepsilon_L^\mu)^* W_{\mu\nu} \varepsilon_L^\nu = -\frac{F_1}{M} + \frac{F_2}{\nu}\frac{1}{M^2}\left(\frac{q_3 M}{Q}\right)^2$$

$$2MW_L = -2F_1 + \frac{2}{M\nu} F_2 \frac{M^2}{Q^2}\left(\nu^2 - q^2\right)$$

$$= -2F_1 + \frac{2M}{\nu} F_2 \left(1 - \frac{\nu^2}{q^2}\right)$$

We are interested in DIS limit

$$Q^2 \to \infty \qquad x = \frac{-q^2}{2p\cdot q} = \frac{Q^2}{2M\nu} = \text{fixed}$$
$$\nu \to \infty$$

$$\frac{2M}{\nu}\left(1 - \frac{\nu^2}{q^2}\right) = \frac{2M}{\nu} - \frac{2M\nu}{q^2}$$

$$= \frac{x}{Q^2/4M^2} + \frac{1}{x}$$

$$2MW_L = \left(\frac{x}{Q^2/4M^2} + \frac{1}{x}\right) F_2 - 2F_1$$

$$\to \frac{1}{x}\left(F_2 - 2xF_1\right)$$

What about $W_T = \frac{1}{2}\sum_T (\varepsilon_T^\mu)^* W_{\mu\nu} \varepsilon_T^\nu = ?$

$$\varepsilon_T = \frac{1}{\sqrt{2}}(0, 1, \pm i, 0)$$

$$\varepsilon_T^\pm \cdot p = 0 \qquad \varepsilon_T^\pm \cdot q = 0$$

$$(\varepsilon_T^\pm)^* \cdot g \cdot \varepsilon_T^\pm = \frac{1}{2}\cdot(-1-1) = -1$$

$$M W_T = -M \cdot 2 \cdot \frac{-F_1}{M}$$

$$\cancel{= +4F_1}$$
$$= 2F_1$$

$$R = \frac{\sigma_L}{\sigma_T} = \frac{\frac{1}{x}(F_2 - 2xF_1)}{2F_1} = \frac{F_2 - 2xF_1}{2xF_1}$$

In the DIS limit, we had worked out the structure functions as

$$F_1(x, Q^2) = \sum_f \frac{e_f}{2} q_f(x)$$

$$F_2(x, Q^2) = \sum_f e_f^2 \, x \, q_f(x)$$

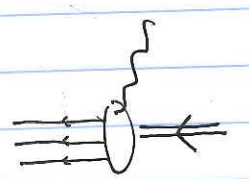
where $q_f(x)$ = quark parton distribution function

What are these distribution functions?

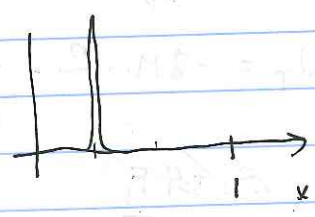
- universal for a given hadron
- describe the distribution of partons inside a hadron.

For the nucleon (proton) there isn't just the uud valence quarks, but also the $\bar{q}q$ pairs and the gluons.

In our simplest parton model, 3 non-interacting valence quarks



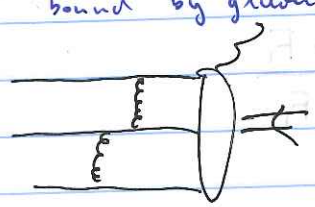
$F_i(x)$



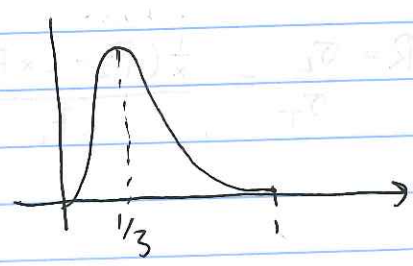
$\delta(x_i - \frac{1}{3})$

$\Downarrow$

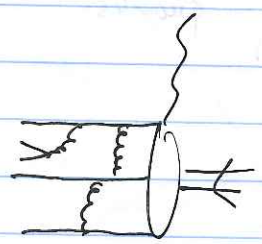
bound by gluons

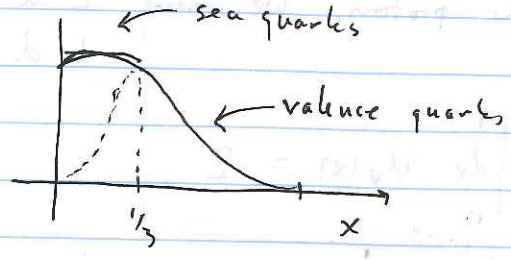


$F_i(x)$



and the glue can split further to $\bar{q}q$ pairs





The gluonic PDFs can not be directly probed w/ electromagnetic current.

The anti-quarks will contribute the same as quarks except

$$q(x) \to \bar{q}(x)$$

$$\Rightarrow \quad F_2(x, Q^2) = 2x F_1(x, Q^2) = \sum_f e_f^2 \, x \left[q_f(x) + \bar{q}_f(x) \right]$$

We can separate the valence and sea quark PDFs

$$q_f(x) = q_{V_f}(x) + q_{S_f}(x)$$

$$\bar{q}_f(x) = \bar{q}_{S_f}(x)$$

These PDFs are long-distance (non-perturbative) structure, and hence, not computable analytically.

There are however, useful constraints (sum-rules) we can infer

$$\int_0^1 dx \left[q_{S_f}(x) - \bar{q}_{S_f}(x) \right] = 0$$

The proton contains no net sea quarks (by definition)

In the proton, we have 2 u (valence) quarks
1 d (valence)

$$\int_0^1 dx\ u_v(x) = 2$$

$$\int_0^1 dx\ d_v(x) = 1$$

$$\Rightarrow \int_0^1 dx\ \left[u(x) - \bar{u}(x)\right] = 2$$

$$\int_0^1 dx\ \left[d(x) - \bar{d}(x)\right] = 1$$

$$\int_0^1 dx\ \left[s(x) - \bar{s}(x)\right] = 0$$

$$\vdots$$

Momentum conservation

$$\sum_f \int_0^1 dx\ x\left[q_f(x) + \bar{q}_f(x)\right] + \int_0^1 dx\ x\ G(x) = 1$$

$G(x)$ = gluon PDF

Empirically (low Q^2 + simple parton model)

$$\int_0^1 dx\ u(x)\ x \sim 0.3 \qquad \int dx\ x\,G(x) \sim 0.5$$

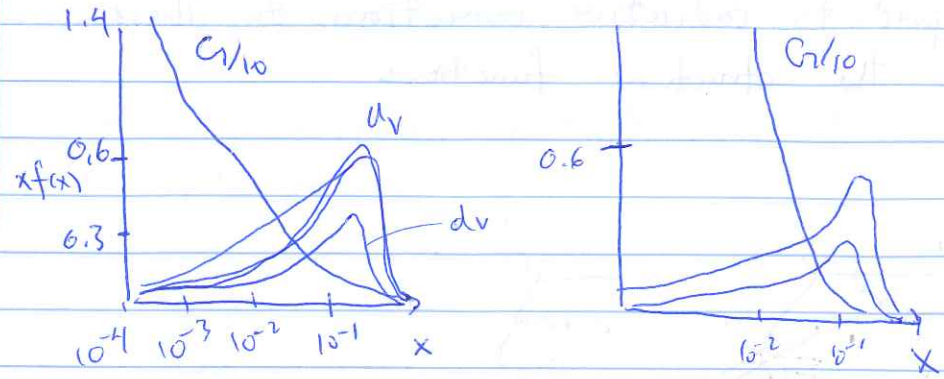
$$\int dx\ d(x)\ x \sim 0.15$$

$$\int dx\ \bar{u}(x)\ x \sim .03$$

$$\int dx\ \bar{d}(x)\ x \sim .015$$

– Half the proton momentum is glue!

– Clearly different from NR Quark Model

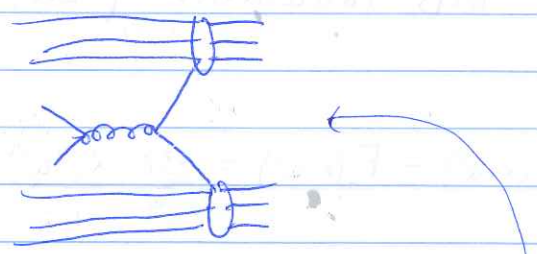


$Q^2 = 20\ GeV^2$ $\qquad$ $Q^2 = 10{,}000\ GeV^2$

This is also why LHC pp collisions w/ 7 TeV beams only have a 'reach' of ~3 TeV

COM, $s = 14$ TeV

but in these collisions, the partons only carry a fraction of the total momentum of the protons



These PDFs are the same for

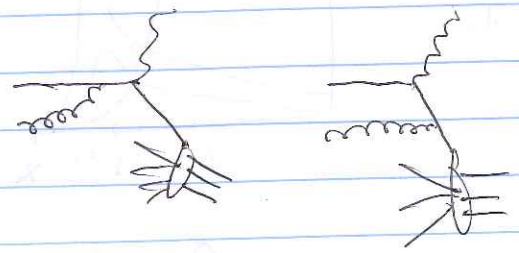
ep scattering

νp scattering

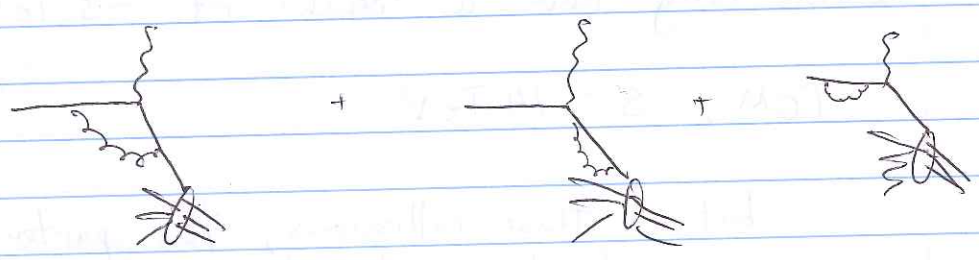
pp scattering (Drell-Yan production of $l\bar{l}$)

They are important for understanding QCD, LHC, etc.

QCD predicts radiative corrections to these PDFs & the structure functions



These contain IR singularities (just like QED)
To have IR safe quantity, we must add all diagrams at same order

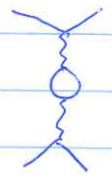


A detailed QCD calculation (Zee, Wilczek, Treiman PRD 10 1974)

$$2x\,F_1(x,Q^2) - F_2(x,Q^2) = \frac{-32}{9}\,\frac{\alpha_s(Q^2)}{4\pi}\,F_2(x,Q^2)$$

What is $\alpha_s(Q^2)$?

We know from QED, it is determined from vacuum polarization



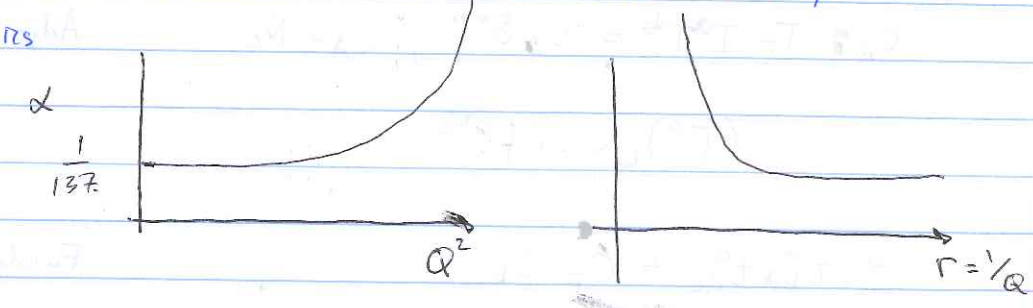
$$\Rightarrow \quad e^2(Q) = e_0^2 \left(\frac{1}{1 - \Pi(q^2)} \right)$$

$$\Rightarrow \quad \alpha_{f.s.}(Q^2) = \frac{\alpha_{f.s.}(Q_0^2)}{1 - \frac{\alpha(Q_0^2)}{3\pi} \ln \frac{Q^2}{Q_0^2}}$$

For assymptotically high Q^2 (short distance), the charge get's larger, and as

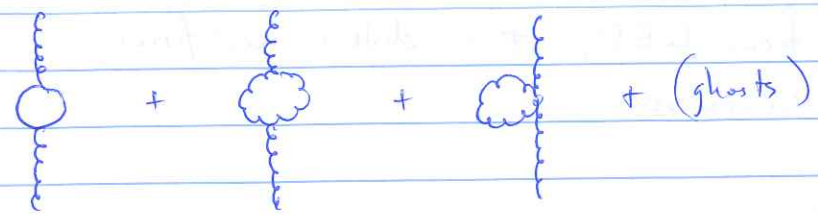
$$\frac{\alpha(Q_0^2)}{3\pi} \ln \frac{Q^2}{Q_0^2} \longrightarrow 1 \text{ , the charge diverges.}$$

This is the 'Landau Pole', and the reason Landau didn't believe QFT was the correct description of physics



We can define the electric charge @ $Q^2 = m_e^2 = Q_0^2$

You know the story for QCD is different



$$\Rightarrow \alpha_s(\mu) = \frac{\alpha_s(Q_0^2)}{1 - \frac{\alpha_s(Q_0^2)}{3\pi}\left[\frac{n_f}{2} - \frac{11}{4}C_A\right]\ln\frac{\mu^2}{Q_0^2}}$$

$$= \frac{\alpha_s(Q_0^2)}{1 + \frac{\alpha_s(Q_0^2)}{4\pi}\beta_0 \ln\frac{\mu^2}{Q_0^2}}$$

$$\beta_0 = \frac{11 C_A - 2n_f}{3}$$

$$C_A \, \mathrm{Tr}\, T^a T^b = C_A \delta^{ab}, \quad C_A = N_c$$ Adjoint

$$(T^a)_{bc} = -i f^{abc}$$

$$\sum_a t^a_{bd} t^a_{dc} = C_F \delta_{bc}$$ Fundamental

$$C_F = \frac{N_c^2 - 1}{2N_c}$$

$$C_F(N_c=3) = \frac{4}{3}$$

$$C_A(N_c=3) = 3$$

$$\beta_0(N_c=3) = \frac{33-2n_f}{3}$$

Note, there is some value of Q^2 where this diverges

$$1 = \frac{\alpha_s(Q_0^2)}{4\pi}\beta_0 \ln\frac{Q_0^2}{\mu^2}$$

$$\cancel{\frac{1-\frac{\alpha_s(Q_0^2)}{4\pi}\beta_0 \ln Q_0^2}{\frac{\alpha_s(Q_0^2)}{4\pi}} = -\ln\mu^2}$$

$$\cancel{\Lambda_{QCD}^2 =}$$

$$\frac{4\pi}{\alpha_s(Q_0^2)\beta_0} = -\ln\frac{\mu^2}{Q_0^2}$$

$$\mu^2 = Q_0^2 e^{-4\pi/\alpha_{s,0}\beta_0}$$

$$\Lambda_{QCD}^2 \equiv Q_0^2 e^{-4\pi/\alpha_{s,0}\beta_0}$$

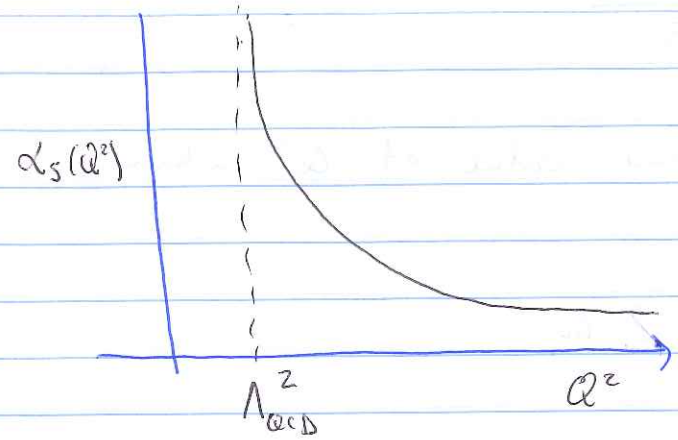
$$\Rightarrow \quad \alpha_s(Q^2) = \frac{4\pi}{\beta_0 \ln Q^2/\Lambda_{QCD}^2}$$

Dimensional Transmutation

Note: $Q^2 \to \infty$ $\quad \lim_{Q^2\to\infty} \alpha_s(Q^2) = 0$

Exactly what we expect, the strength of the qq coupling goes to 0 at assymptotically high Q^2 ($r=0$)

But, we see at low energy, $\alpha_s(Q^2)$ grows



Suppose $m_q = 0$ for all q

$\mathcal{L}_{QCD}$ is scale invariant

$$\mathcal{L}_{QCD} = \bar{\psi}_q \, i \not{D} \, \psi_q$$

But the interactions of the theory generate a scale, Λ_{QCD}!

Note: Λ_{QCD} depends upon which loop order you work to (it is not observable)

But it gives a nice qualitative way to understand the generation of mass in QCD.

$$\Rightarrow m_p = \# \Lambda_{QCD} + \#' m_q + \cdots$$

$$\lim_{m_u, m_d \to 0} m_p \sim 900 \text{ MeV}$$

But this is only true for

$$33 - 2n_f > 0$$

for a theory w/16 fermions, still QCD like as we know it.

for $n_f \geq 17$, the number of fermion loops is sufficient to make the theory QED like.

$$\cancel{\frac{\partial \alpha_s(Q^2)}{\partial \ln Q^2}} \cancel{<0}$$

$$\frac{\partial}{\partial \ln Q^2} \alpha_s(Q^2) = -\frac{4\pi}{\beta_0}\left(\frac{1}{\ln Q^2/\Lambda^2}\right)^2 < 0$$

negative β function

Politzer & (Gross & Wilczek)
Nobel Prize

vs QED

$$\frac{\partial}{\partial \ln Q^2} \frac{\alpha_0}{1 - \frac{\alpha_0}{3\pi}\ln Q^2/Q_0^2} = \frac{\alpha_0/3\pi}{\left(1 - \frac{\alpha_0}{3\pi}\ln Q^2/Q_0^2\right)^2} > 0$$