Introduction to the Standard Model
William and Mary PHYS 771 Spring 2014

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Class information, including syllabus and homework assignments can be found at
`http://ntc0.lbl.gov/~walkloud/wm/courses/PHYS_771/`
or
`http://cyclades.physics.wm.edu/~walkloud/wm/PHYS_771/`

Homework Assignment 2: due no sooner than Wed., 26 February

1. For a compact Lie group, we discussed the importance of the Cartan subalgebra, H spanned by the maximal set of commuting generators,

$$H_i^\dagger = H_i, \qquad [H_i, H_j] = 0, \qquad \mathrm{Tr}(H_i H_j) = \lambda\delta_{ij}, \qquad H_i|\mu, x, D\rangle = \mu_i|\mu, x, D\rangle, \tag{1}$$

where $?_i$ are the *weights*, x is other information needed to describe the state and D is the representation. We discussed in the adjoint representation, $[T_a]_{bc} = -if_{abc}$, the states can be described by the generators, as the same label describes both the generators (a) and the states (elements of the matrix (b, c)). This led to important properties, such as linear combinations of states correspond to linear combinations of generators

$$a|X_a\rangle + b|X_b\rangle = |aX_a + bX_b\rangle \tag{2}$$

with a convenient scalar product

$$\langle X_a|X_b\rangle = \lambda^{-1}\mathrm{Tr}(X_a^\dagger X_b) \tag{3}$$

which allowed us to determine the action of a generator on a state

$$X_a|X_b\rangle = |[X_a, X_b]\rangle \tag{4}$$

and so the states which correspond to the Cartan subalgebra satisfy

$$H_i|H_j\rangle = 0 \tag{5}$$

while other states corresponding to the rest of the generators satisfy

$$H_i|E_\alpha\rangle = \alpha_i E_\alpha \to [H_i, E_\alpha] = \alpha_i E_\alpha, \qquad [H_i, E_\alpha^\dagger] = -\alpha_i E_\alpha^\dagger \to E_\alpha^\dagger = E_{-\alpha}, \tag{6}$$

and $[E_\alpha, E_{-\alpha}] = \alpha \cdot H$.

In the adjoint representation, these non zero weights α_i are called *roots* and uniquely specify the states. For each non zero pair of root vectors, $\pm\alpha$, there is an $SU(2)$ subalgebra of the group, with generators

$$E_\pm \equiv |\alpha|^{-1}E_{\pm\alpha}, \qquad\qquad E_3 \equiv |\alpha|^{-2}\alpha \cdot H. \tag{7}$$

(a) what are the commutation relations

$$
\begin{aligned}
[E_3, E_\pm] &= ? \\
[E_+, E_-] &= ?
\end{aligned} \tag{8}
$$

(b) in $SU(3)$, what are the explicit commutation relations $[E_+, E_-]$ for all $SU(2)$ subalgebras?

(c) in $SU(3)$, calculate f_{147} and f_{458}.

(d) $SU(2)$ subalgebra of $SU(3)$: 1

i. show that t_1, t_2 and t_3 generate an $SU(2)$ subalgebra of $SU(3)$

ii. take t_3 as the Cartan generator of the subalgebra and the raising lowering operators as $t_\pm = (t_1 \pm i t_2)/\sqrt{2}$. What are the eigenvectors in this representation, $|1\rangle, |2\rangle, |3\rangle$ and their corresponding weight vectors? Which is the lowest?

iii. Is there an invariant subspace in this representation? (do the raising/lowering operators span the space of these eigenvectors?)

(e) same as problem 1d except with t_2, t_5 and t_7

2. We discussed the need for a new quantum number, color, to explain the $|\Delta^{++}\rangle$ and $|\Omega\rangle$ decuplet states, which have totally symmetric spin and flavor wave-functions. This also leads us to require the combined spin and flavor wave-functions of the octet baryons are totally symmetric, which can be constructed for example as

$$
|B\uparrow\rangle = \frac{1}{\sqrt{2}}\left(|\psi_f^S\rangle|\psi_s^S\rangle + |\psi_f^A\rangle|\psi_s^A\rangle\right) \tag{9}
$$

where the mixed-symmetric and mixed-anti-symmetric spin wave-functions are

$$
\begin{aligned}
|\uparrow S\rangle &= \frac{1}{\sqrt{6}}\left(|\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle - 2|\uparrow\uparrow\downarrow\rangle\right) \\
|\uparrow A\rangle &= \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle\right)
\end{aligned} \tag{10}
$$

where the ordering of the labels correspond to quark 1,2 and 3 in the proton wave-function. Similarly, the proton mixed-symmetric and mixed-anti-symmetric flavor wave functions are

$$
\begin{aligned}
|pS\rangle &= \frac{1}{\sqrt{6}}\left(|udu\rangle + |duu\rangle - 2|uud\rangle\right) , \\
|pA\rangle &= \frac{1}{\sqrt{2}}\left(|ud\rangle - |du\rangle\right)|u\rangle \\
&= \frac{1}{\sqrt{2}}\left(|udu\rangle - |duu\rangle\right) ,
\end{aligned} \tag{11}
$$

where the total proton spin-up spin-flavor wave-function is

$$
\begin{aligned}
|p\uparrow\rangle = \frac{1}{\sqrt{18}}\Big[& \ |uud\rangle\left(|\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle - 2|\uparrow\uparrow\downarrow\rangle\right) \\
& + |udu\rangle\left(|\uparrow\uparrow\downarrow\rangle + |\downarrow\uparrow\uparrow\rangle - 2|\uparrow\downarrow\uparrow\rangle\right) \\
& + |duu\rangle\left(|\uparrow\downarrow\uparrow\rangle + |\uparrow\uparrow\downarrow\rangle - 2|\downarrow\uparrow\uparrow\rangle\right)\Big]
\end{aligned} \tag{12}
$$

and recall the ordering of the labels is correlated

$$|uud\rangle \otimes |\uparrow\downarrow\uparrow\rangle = |uud\rangle|\uparrow\downarrow\uparrow\rangle = |u\uparrow, u\downarrow, d\uparrow\rangle\,,$$
$$\langle uud|\langle\uparrow\downarrow\uparrow|\;|uud\rangle|\uparrow\uparrow\downarrow\rangle = 0 \tag{13}$$

This is not the only way to construct an anti-symmetric wave-function for the octet baryons. For example

$$|p\uparrow\rangle_A = \frac{1}{\sqrt{2}}\left(|pA\rangle|\uparrow S\rangle - |pS\rangle|\uparrow A\rangle\right) \tag{14}$$

is a totally anti-symmetric wave-function, without the need for color.

(a) following Eq. (14), determine the neutron spin-flavor wave-function in full detail as in Eq. (12)

(b) compute the proton magnetic moment with Eq. (14) as we did in class

(c) what is

$$\frac{\mu_n}{\mu_p} =? \tag{15}$$

with these anti-symmetric, colorless wave-functions? How does it compare with experiment?

3. In class, we determined the hyper-fine quark-quark interaction Hamiltonian for the $N-\Delta$ and $\Lambda-\Sigma$ systems;

$$H_{S\cdot S'} = \frac{2}{3}\sum_{j\neq k}\kappa_j\kappa_k\vec{S}_j\cdot\vec{S}_k\,, \qquad\qquad \kappa_j = \frac{\kappa}{m_j} \tag{16}$$

where κ is a constant proportional to the quark chromo-magnetic moment, with mass dimension 3/2 and m_j is the constituent quark mass for quark flavor j (the strong interactions are flavor blind so κ is the same for all quarks). We worked out

$$H^{\Lambda\Sigma}_{S\cdot S'} = \frac{2}{3}\left[\kappa_l^2\begin{pmatrix}1/4\\1/4\\-3/4\end{pmatrix} + \kappa_l\kappa_s\begin{pmatrix}1/2\\-1\\0\end{pmatrix}\right] \text{ for } \begin{cases}\Sigma^* & S_d=1 \;\; S_T=3/2\\ \Sigma & S_d=1 \;\; S_T=1/2\\ \Lambda & S_d=0 \;\; S_T=1/2\end{cases} \tag{17}$$

where $\kappa_u = \kappa_d = \kappa_l$ for the light quarks and S_d is the spin of the light di-quark and S_T is the total spin.

(a) determine the equivalent expression, Eq. (17) for the Ξ-Ξ^* system.

(b) determine $H_{S\cdot S'}$ for the spin-0 and spin-1 mesons

(c) assuming that κ in the baryons is the same as the mesons, relate $m_\Delta - m_N$ to $m_\rho - m_\pi$ and compare to the experimental values