Introduction to the Standard Model
William and Mary PHYS 771 Spring 2014

Instructor: André Walker-Loud, walkloud@wm.edu
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Class information, including syllabus and homework assignments can be found at
`http://ntc0.lbl.gov/~walkloud/wm/courses/PHYS_771/`
or
`http://cyclades.physics.wm.edu/~walkloud/wm/PHYS_771/`

Homework Assignment 3: due Friday 28 March

1. **[20 pts.]** Consider two fermions (e, μ) and a complex scalar coupled through $U(1)$ gauge symmetry

$$\mathcal{L} = \sum_{f=e,\mu} \bar{\psi}_f \left[i\not{D} - m_f \right] \psi_f + [D_\mu \phi]^\dagger [D^\mu \phi] - m_\phi^2 \phi^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{1}$$

with $D_\mu = \partial_\mu + iqA_\mu$.

   (a) compute the unpolarized (spin-averaged) differential cross section for $e^+e^- \to \mu^+\mu^-$ in the limit $m_e = 0$ in the center of mass frame

$$\frac{d\sigma}{d\cos\theta} = ? \tag{2}$$

   [8 pts.] *Solution:* see attached notes

   (b) same as 1a but for $e^+e^- \to \phi^+\phi^-$
   [10 pts.] *Solution:* see attached notes

   (c) compare these cross sections in the relativistic limit (ignore masses of final states as well)
   [2 pts.] *Solution:* see attached notes

2. **[25 pts.]** Pion decay: the pion decay amplitude is proportional to

$$M_{\pi^- \to \ell\nu_\ell} \propto f_\pi q^\mu \bar{u}_\ell \gamma_\mu (1 - \gamma_5) v_{\bar{\nu}_\ell} \,. \tag{3}$$

Recalling the $(A \to 1 + 2)$ decay rate is given in the rest frame of A by

$$\Gamma = \frac{|\vec{p}_f|}{32\pi^2 M_A^2} \int d\Omega |M|^2 \tag{4}$$

   (a) calculate the ratio

$$R = \frac{\Gamma(\pi^- \to e^- \bar{\nu}_e)}{\Gamma(\pi^- \to \mu^- \bar{\nu}_\mu)} \tag{5}$$

   and compare with the PDG value
   [15 pts.] *Solution:* see attached notes

(b) suppose the weak decay amplitude were instead given by

$$\tilde{M}_{\pi^- \to \ell \nu_\ell} \propto g_\pi \bar{u}_\ell \gamma_5 v_{\bar{\nu}_\ell} \,, \tag{6}$$

compute the corresponding ratio R for this pseudo-scalar model

[10 pts.] *Solution:* see attached notes

3. **[30 pts.]** In class, we discussed the complex ϕ^4 theory with spontaneous symmetry breaking. We worked in 'polar' coordinates with $\phi = \frac{\rho+v}{\sqrt{2}} e^{i\theta}$. We computed the one-loop s-channel contribution to $\theta\theta \to \theta\theta$ scattering amplitude with intermediate ρ particles, using dimensional regularization, finding

$$\delta A(s) = \frac{1}{2}\left(\frac{s}{4\pi v^2}\right)^2 \left[\frac{1}{\epsilon} - \gamma_E + \ln(4\pi) - \ln\frac{m^2}{\mu^2} - 2 + \begin{cases} i\sigma(s)(2\tan^{-1}(i\sigma(s)) - \pi) & s < 4m^2 \\ \sigma(s)\ln\frac{1+\sigma(s)}{1-\sigma(s)} - i\pi\sigma(s) & s > 4m^2 \end{cases}\right. \tag{7}$$

with $\sigma(s) = \sqrt{1 - 4m^2/s}$.

(a) Compute the t and u channel contributions to this scattering amplitude and combine them with this s-channel result

[10 pts.] *Solution:* see attached notes

(b) to renormalize the theory, we must add a higher dimensional operator, with four derivatives and four θ fields. Write down the form of this operator.

[5 or 10 pts.: correlated with next problem] *Solution:* see attached notes

(c) use this new operator to act as the counterterm to renormalize the scattering amplitude. What is the $1/\epsilon$ contribution to this operator?

[10 or 5 pts.: correlated with previous problem] *Solution:* see attached notes

(d) by requiring the renormalized amplitude to be independent of the dimensional-regularization scale μ, determine the μ dependence of the coefficient of the new operator

[5 pts.] *Solution:* see attached notes

4. **[10 pts.]** Suppose we have a charge distribution $\rho(r) = Ne^{-mr}$ with $\int d^3r\rho(r) = 1$.

(a) what is $N =$?

[2 pts.] *Solution:* see attached notes

(b) what is the resulting form factor for Compton scattering

$$F(\vec{q}) = \int d^3r e^{i\vec{q}\cdot\vec{r}} \rho(\vec{r}) = ? \tag{8}$$

[5 pts.] *Solution:* see attached notes

(c) Given the Taylor expansion of the form factor

$$F(q) = 1 - \frac{1}{6} r^2 q^2 + \ldots \tag{9}$$

What is the predicted 'charge-radius' of the proton, using the electric form factor, $G_E(-q^2) = (1 - q^2/m_0^2)^{-2}$ with $m_0^2 = 0.71$ GeV^2? Compare with the PDG value of the charge radius and the recent determination from muonic-Hydrogen (use Google to find these values).

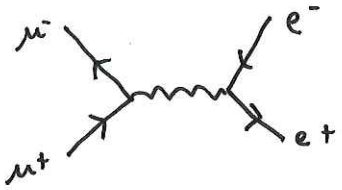
[**3 pts.**] *Solution:* see attached notes

HW 3

1) $$\mathcal{L} = \sum_{f=\{e,\mu\}} \bar{\psi}_f [i \not{D} - m_f] \psi_f + [D_\mu \phi]^\dagger [D^\mu \phi] - m_\phi^2 \phi^\dagger \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$D_\mu = \partial_\mu + i q A_\mu$$

a) spin-averaged differential cross section for $e^+e^- \to \mu^+\mu^-$



, $P = p_{e^-} + p_{e^+} = p_{\mu^-} + p_{\mu^+}$

$$iA = \bar{u}(p_{\mu^-})(+iq_\mu \gamma^\mu) v(p_{\mu^+}) \frac{-i g_{\mu\nu}}{P^2 + i\varepsilon} \bar{v}(p_{e^+})(+iq_e \gamma^\nu) u(p_{e^-})$$

$$= \frac{i q_\mu q_e}{P^2} \bar{u}(p_{\mu^-}) \gamma^\mu v(p_{\mu^+}) \bar{v}(p_{e^+}) \gamma_\mu u(p_{e^-}) \quad , \quad \text{let } q_\mu = q_e = -e$$

To get the spin-averaged cross-section, we

$$\frac{1}{4} \sum_{rr'} \sum_{ss'} |A|^2$$

$$|A|^2 = \frac{e^4}{(P^2)^2} \bar{u}(p_{\mu^-}) \gamma^\mu v(p_{\mu^+}) \bar{v}(p_{\mu^+}) \gamma^\nu u(p_{\mu^-}) \cdot \bar{v}(p_{e^+}) \gamma_\mu u(p_{e^-}) \bar{u}(p_{e^-}) \gamma_\nu v(p_{e^+})$$

$$\frac{1}{4} \sum_{rr'} \sum_{ss'} |A|^2 = \frac{e^4}{4(P^2)^2} \langle \bar{u}(p_{\mu^-}) \gamma^\mu v(p_{\mu^+}) \bar{v}(p_{\mu^+}) \gamma^\nu u(p_{\mu^-}) \rangle \langle \bar{v}(p_{e^+}) \gamma_\mu u(p_{e^-}) \bar{u}(p_{e^-}) \gamma_\nu v(p_{e^+}) \rangle$$

$$= \frac{e^4}{4(P^2)^2} \langle (\not{p}_{\mu^-} + m_\mu) \gamma^\mu (\not{p}_{\mu^+} - m_\mu) \gamma^\nu \rangle \langle (\not{p}_{e^+} - m_e) \gamma_\mu (\not{p}_{e^-} + m_e) \gamma_\nu \rangle$$

We are ignoring m_e

$$\frac{1}{4}\sum_{\text{spins}} |A|^2 \cong \frac{e^4}{4s^2} \langle (\not{p}_{\mu-} + m)\gamma^\mu (\not{p}_{\mu+} - m)\gamma^\nu \rangle \langle \not{p}_{e+}\gamma_\mu \not{p}_{e-}\gamma_\nu \rangle$$

The electron trace term gives

$$\langle \not{p}_{e+}\gamma_\mu \not{p}_{e-}\gamma_\nu \rangle = 4\left(g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\beta} g_{\mu\nu} + g_{\alpha\nu} g_{\beta\mu}\right) p_{e+}^\alpha p_{e-}^\beta$$

$$= 4\left(p_{e+\mu} p_{e-\nu} + p_{e+\nu} p_{e-\mu} - p_{e-}\cdot p_{e+}\, g_{\mu\nu}\right)$$

COM: $p_{e-}\cdot p_{e+} = E_e^2 - \vec{p}_{e-}\cdot\vec{p}_{e+}$

$$= 2E_e^2 \quad (m_e = 0)$$

The muon trace term has two contributions, one similar to the electrons

$$p_{\mu-}\cdot p_{\mu+} = E_\mu^2 - \vec{p}_{\mu-}\cdot\vec{p}_{\mu+}$$

$$= m_\mu^2 + p_\mu^2 + p_\mu^2$$

$$= m_\mu^2 + 2|p_\mu|^2$$

$$\langle \not{p}_{\mu-}\gamma^\mu \not{p}_{\mu+}\gamma^\nu \rangle = 4\left(p_{\mu+}^\mu p_{\mu-}^\nu + p_{\mu+}^\nu p_{\mu-}^\mu - p_{\mu-}\cdot p_{\mu+}\, g^{\mu\nu}\right)$$

$$-\langle m_\mu\gamma^\mu m_\mu\gamma^\nu \rangle = -m_\mu^2\, 4 g^{\mu\nu}$$

$$\Rightarrow \frac{1}{4}\sum_{\text{spins}} |A|^2 = \frac{4e^4}{s^2}\left[p_{\mu+}^\mu p_{\mu-}^\nu + p_{\mu+}^\nu p_{\mu-}^\mu - \underbrace{(m_\mu^2 + 2|p_\mu|^2)}_{2}\, g^{\mu\nu}\right]\left[p_{e+\mu} p_{e-\nu} + p_{e+\nu} p_{e-\mu} - 2E_e^2 g_{\mu\nu}\right]$$

$$= \frac{4e^4}{s^2}\Big[2\, p_{\mu+}\cdot p_{e+}\, p_{\mu-}\cdot p_{e-} + 2\, p_{\mu+}\cdot p_{e-}\, p_{\mu-}\cdot p_{e-} - 4E_e^2 (p_{\mu+}\cdot p_{\mu-}) + 16 E_e^2 \underbrace{(m_\mu^2 + |p_\mu|^2)}_{E_\mu^2}$$

$$- 2E_\mu^2 \cdot 2(p_{e+}\cdot p_{e-})^2 \Big]$$

$$-2E_e^2\, p_{\mu+}\cdot p_{\mu-} + 8E_e^2 E_\mu^2 - 2E_\mu^2\, p_{e+}\cdot p_{e-}$$

$$p_{\mu+}\cdot p_{\mu-} = E_\mu^2 - \vec{p}_{\mu+}\cdot\vec{p}_{\mu-} \qquad p_{e+}\cdot p_{e-} = E_e^2\cdot 2$$

$$= E_\mu^2 + |p_\mu|^2$$

$$= m_\mu^2 + 2|p_\mu|^2$$

$$= -p_{e-}\cdot p_{e+}\left(m_\mu^2 + 2|p_\mu|^2\right) + 4p_{e+}\cdot p_{e-}\left(m_\mu^2 + |p_\mu|^2\right) - 2p_{e+}\cdot p_{e-}\left(m_\mu^2 + |p_\mu|^2\right)$$

$$= p_{e+}\cdot p_{e-}\left[-m_\mu^2 + 4m_\mu^2 - 2m_\mu^2 - 2|p_\mu|^2 + 4|p_\mu|^2 - 2|p_\mu|^2\right]$$

$$= m_\mu^2\, p_{e+}\cdot p_{e-}$$

$$= 2m_\mu^2 E_e^2$$

$$\Rightarrow \frac{1}{4}\sum_{\text{spins}} |A|^2 = \frac{8e^4}{s^2}\left[p_{\mu+}\cdot p_{e+}\, p_{\mu-}\cdot p_{e-} + p_{\mu+}\cdot p_{e-}\, p_{\mu-}\cdot p_{e+} + 2m_\mu^2 E_e^2\right]$$

$$p_{\mu+}\cdot p_{e+} = E^2 - E|p_\mu|\cos\theta \qquad s = 4E^2$$

$$p_{\mu-}\cdot p_{e-} = E^2 - E|p_\mu|\cos\theta$$

$$p_{\mu+}\cdot p_{e-} = E^2 + E|p_\mu|\cos\theta$$

$$p_{\mu-}\cdot p_{e+} = E^2 + E|p_\mu|\cos\theta$$

$$= \frac{8e^4}{16E^4}\left[\left(E^2 - E|p_\mu|\cos\theta\right)^2 + \left(E^2 + E|p_\mu|\cos\theta\right)^2 + 2m_\mu^2 E^2\right]$$

$$= \frac{e^4}{2}\left[2 + 2\frac{|p_\mu|^2}{E^2}\cos^2\theta + 2\frac{m_\mu^2}{E^2}\right]$$

$$= e^4\left[1 + \frac{m_\mu^2}{E^2} + \left(1 - \frac{m_\mu^2}{E^2}\right)\cos^2\theta\right]$$

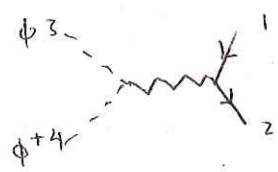
The differential cross section is given by

$$\frac{d\sigma}{d\Omega} = \frac{1}{2E_{cm}^2}\frac{|p_\mu|}{16\pi^2 E_{cm}}\frac{1}{4}\sum_{spins}|A|^2 \quad , \qquad E_{cm} = 2E$$

$$= \left(\frac{e^2}{4\pi}\right)^2 \frac{1}{8E^2}\frac{|p_\mu|}{2E}\left[1+\frac{m_\mu^2}{E^2}+\left(1-\frac{m_\mu^2}{E^2}\right)\cos^2\theta\right]$$

$$= \frac{\alpha}{16E^2}\sqrt{1-\frac{m_\mu^2}{E^2}}\left[1+\frac{m_\mu^2}{E^2}+\left(1-\frac{m_\mu^2}{E^2}\right)\cos^2\theta\right]$$

b] What is the differential cross section for $e^+e^- \to \phi^+\phi$?



$$(D_\mu\phi)^\dagger D^\mu\phi = \left[(\partial_\mu + iqA_\mu)\phi\right]^\dagger\left[(\partial^\mu + iqA^\mu)\phi\right]$$
$$= \partial_\mu\phi^\dagger\, iqA^\mu\phi - iqA_\mu\phi^\dagger\partial^\mu\phi + \cdots$$

$$iA = iq_\phi(-i)(+ip_4 - ip_3)_\mu \frac{-ig^{\mu\nu}}{s}\bar{v}(p_2)(iq\gamma_\nu)u(p_1) \quad , \quad q_\phi = e$$

$$= (i)^5(-)^2 e(-e)(p_4-p_3)_\mu \frac{1}{s}\bar{v}(p_2)\gamma^\mu u(p_1)$$

$$A = -\frac{e^2(p_4-p_3)_\mu \bar{v}(p_2)\gamma^\mu u(p_1)}{s}$$

$$|A|^2 = \frac{e^4}{s^2}(p_4-p_3)_\mu(p_4-p_3)_\nu\, \bar{v}(p_2)\gamma^\mu u(p_1)\bar{u}(p_1)\gamma^\nu v(p_2)$$

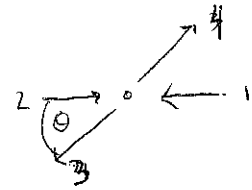
$$\frac{1}{2}\sum_{spins}|A|^2 = \frac{e^4}{2s^2}(p_4-p_3)_\mu(p_4-p_3)_\nu\langle(\not{p}_\nu - m_e)\gamma^\mu(\not{p}+m_e)\gamma^\nu\rangle$$

ignore m_e $\quad \langle \not{p}_2\gamma^\mu\not{p}_1\gamma^\nu\rangle = 4p_{2\alpha}p_{1\beta}\left(g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu} + g^{\alpha\nu}g^{\beta\mu}\right)$

$$= 4\left(p_2^\mu p_1^\nu + p_2^\nu p_1^\mu - p_1\cdot p_2\, g^{\mu\nu}\right)$$

$$\frac{1}{2}\sum_{\text{spins}} |A|^2 = \frac{2e^4}{s^2} P_{(4-3)\mu} P_{(4-3)\nu} \left[P_1^\mu P_2^\nu + P_1^\nu P_2^\mu - 2E^2 g^{\mu\nu} \right]$$

$$(P_{4-3})\cdot(P_{4-3}) = P_4^2 + P_3^2 - 2P_3\cdot P_4$$
$$= 2m_\phi^2 - 2(E^2 + |P_\phi|^2)$$
$$= -4|P_\phi|^2$$



$$P^\mu_{(4-3)} P_{1\mu} = P_4\cdot P_1 - P_3\cdot P_1$$
$$= E^2 - E|P_\phi|\cos\theta$$
$$- E^2 - E|P_\phi|\cos\theta$$
$$= -2E|P_\phi|\cos\theta$$

$$P^\mu_{(4-3)} P_{2\mu} = P_4\cdot P_2 - P_3\cdot P_2$$
$$= +2E|P_\phi|\cos\theta$$

$$= \frac{2e^4}{s^2}\left[-4E^2|P_\phi|^2\cos^2\theta \cdot 2 - 2E^2\cdot(-4|P_\phi|^2) \right]$$

$$= \frac{16e^4}{s^2}\left[E^2|P_\phi|^2(1-\cos^2\theta) \right] \quad , \quad s = 4E^2$$

$$= \frac{e^4}{E^4} E^2|P_\phi|^2 \sin^2\theta$$

$$\frac{1}{2}\sum_{\text{spins}} |A|^2 = e^4\left(1 - \frac{m_\phi^2}{E^2}\right)\sin^2\theta$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{1}{2E_{cm}^2}\frac{|P_\phi|}{16\pi^2 E_{cm}}\frac{1}{2}\sum_{\text{spins}}|A|^2$$

$$= \frac{1}{8E^2}\left(\frac{e^2}{4\pi}\right)^2 \frac{|P_\phi|}{2E_{cm}}\left(1 - \frac{m_\phi^2}{E^2}\right)\sin^2\theta$$

$$= \frac{\alpha^2}{16E^2}\left(1 - \frac{m_\phi^2}{E^2}\right)^{3/2}\sin^2\theta$$

c] In the relativistic limit

$$\frac{d\sigma_{e^+e^- \to \mu^+\mu^-}}{d\Omega} \simeq \frac{\alpha^2}{16E^2}\left(1+\cos^2\theta\right)$$

$$\frac{d\sigma_{e^+e^- \to \phi^+\phi^-}}{d\Omega} \simeq \frac{\alpha^2}{16E^2}\sin^2\theta = \frac{\alpha^2}{16E^2}\left(1-\cos^2\theta\right)$$

This different angular dependence helped resolve the nature of fundamental hadronic constituents, showing they were spin-$\frac{1}{2}$ fermions.

2] π decay

$$M_{\pi^- \to \ell \nu} \propto f_\pi \, q^\mu \, \bar{u}_\ell \gamma_\mu (1-\gamma_5) v_{\nu_\ell}$$

$$\Gamma(\pi \to \ell \nu) = \frac{|\vec{p}_f|}{32\pi^2 m_\pi^2} \int d\Omega \, |M|^2$$

$$|M|^2 \propto f_\pi^2 \sum_{\text{spin}} \bar{u}_\ell \gamma_\mu (1-\gamma_5) v_\nu \bar{v}_\nu (1+\gamma_5) \gamma_\nu u \, q^\mu q^\nu$$

$$= f_\pi^2 q^\mu q^\nu \, \mathrm{Tr}\left[(\not{p}_\ell + m_\ell) \gamma_\mu (1-\gamma_5) \not{p}_\nu (1+\gamma_5) \gamma_\nu \right]$$

$$= f_\pi^2 q^\mu q^\nu \, \mathrm{Tr}\left[(\not{p}_\ell + m_\ell) \gamma_\mu \, 2(1-\gamma_5) \not{p}_\nu \gamma_\nu \right]$$

$$= 2 f_\pi^2 q^\mu q^\nu \Big[\mathrm{Tr}[\not{p}_\ell \gamma_\mu \not{p}_\nu \gamma_\nu] - \underbrace{\mathrm{Tr}[\not{p}_\ell \gamma_\mu \gamma_5 \not{p}_\nu \gamma_\nu]}_{\propto \varepsilon_{\alpha\mu\beta\nu} \;\Rightarrow\; 0 \text{ because contracted with } q^\mu q^\nu} + \underbrace{\mathrm{Tr}[m_\ell \gamma_\mu \not{p}_\nu \gamma_\nu]}_{=0 \text{ from } \mathrm{Tr}[\gamma^{odd}]} \Big]$$

$$\underbrace{-\mathrm{Tr}[m_\ell \gamma_\mu \gamma_5 \not{p}_\nu \gamma_\nu]}_{=0 \text{ from } \mathrm{Tr}[\gamma^{odd}]}$$

$$= 2 f_\pi^2 q^\mu q^\nu p_\ell^\alpha p_\nu^\beta \cdot 4 \left(g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\beta} g_{\mu\nu} + g_{\alpha\nu} g_{\mu\beta} \right)$$

$$= 8 f_\pi^2 \left[2 \, p_\ell \cdot q \, p_\nu \cdot q - p_\ell \cdot p_\nu \, q^2 \right]$$

We can work in the π rest frame, $q^\mu = (m_\pi, \vec{0})$

$$\cancel{\Rightarrow |M|^2_{cm} \propto 8 f_\pi^2 \left[2 m_\pi^2 E_\ell E_\nu - m_\pi^2 E_\ell E_\nu + m_\pi^2 |p_F|^2 \right]}$$

π - rest frame

$$|M|^2_{cm} \propto f_\pi^2 \left[2 m_\pi^2 E_\ell E_\nu - m_\pi^2 (E_\nu E_\ell - \vec{p}_\ell \cdot \vec{p}_\nu) \right]$$

$$= m_\pi^2 f_\pi^2 E_\nu \left[2E_\ell - E_\ell \mp |\vec{p}_\ell| \right]$$

$$= m_\pi^2 f_\pi^2 E_\nu \left[E_\ell - |\vec{p}_\ell| \right]$$

Conservation of four momentum

$$m_\pi = E_\ell + E_\nu$$

$$E_\ell^2 = m_\pi^2 + E_\nu^2 - 2 m_\pi E_\nu$$

$$E_\ell^2 = m_\ell^2 + p_\ell^2$$

$$= m_\ell^2 + E_\nu^2$$

$$E_\nu = \frac{m_\pi^2 - m_\ell^2}{2 m_\pi}$$

$$\Rightarrow |M|^2_{cm} \propto m_\pi^2 f_\pi^2 \frac{m_\pi^2 - m_\ell^2}{2 m_\pi} \left[E_\ell - |p_\ell| \right]$$

We see there is no angular dependence, so the solid angle integral cancels in the ratio $\dfrac{\Gamma(\pi \to e\nu)}{\Gamma(\pi \to \mu\nu)}$

$$R = \frac{|M|^2_{cm} |p_e|}{|M_{\pi\to\mu}|^2_{cm} |p_\mu|} = \frac{m_\pi^2 - m_e^2}{m_\pi^2 - m_\mu^2} \frac{|E_e - p_e|}{|E_\mu - p_\mu|} \frac{E_{\nu_e}}{E_{\nu_\mu}}, \quad E_\ell - p_\ell = E_\ell - E_\nu = m_\pi - 2E_\nu$$

$$= \left(\frac{1 - m_e^2/m_\pi^2}{1 - m_\mu^2/m_\pi^2} \right)^2 \frac{m_e^2/m_\pi^2}{m_\mu^2/m_\pi^2}$$

$$= \frac{m_e^2}{m_\mu^2} \left(\frac{1 - m_e^2/m_\pi^2}{1 - m_\mu^2/m_\pi^2} \right)^2 \sim 1.2 \times 10^{-4} \quad \left[\begin{array}{c} \text{PDG} \\ 1.230(4) \times 10^{-4} \end{array} \right]$$

b) If the weak amplitude were instead given by

$$M_{\pi \to \ell\nu} \propto g_\pi \, \bar{u}_\ell \gamma_5 v_\nu$$

$$\Rightarrow |M|^2 \propto g_\pi^2 \, \mathrm{Tr}\left[(\not{p}_\ell + m_\ell)\gamma_5 \not{p}_\nu \gamma_5\right]$$

$$\cancel{= -g_\pi^2 \, \mathrm{Tr}\left[\not{p}_\ell \gamma_5 \not{p}_\nu + m_\ell \gamma_5 \not{p}_\nu\right]}$$

$$= -g_\pi^2 \, \mathrm{Tr}\left[\not{p}_\ell \gamma_5^2 \not{p}_\nu + m_\ell \gamma_5^2 \not{p}_\nu\right]$$

$$= -g_\pi^2 \, \mathrm{Tr}\left[\not{p}_\ell \not{p}_\nu + m_\ell \not{p}_\nu\right]$$

$$= -g_\pi^2 \, 4 \, p_\ell \cdot p_\nu$$

$$= -4 g_\pi^2 \, m_\pi E_\nu \quad [\pi \text{ rest frame}]$$

Again, the angular dependence is trivial so

$$R = \frac{m_\pi E_{\nu_e} |p_{\nu_e}|}{m_\pi E_{\nu_\mu} |p_{\nu_\mu}|} = \frac{E_{\nu_e}^2}{E_{\nu_\mu}^2}$$

$$= \left(\frac{1 - m_e^2/m_\pi^2}{1 - m_\mu^2/m_\pi^2}\right)$$

$$\simeq 5.5$$

This pseudoscalar decay model was a competitive weak interaction model very early on, but is clearly ruled out by comparing w/ experimental results.

3] complex ϕ^4 theory with spontaneous symmetry breaking:
$\theta\theta$ scattering @ 1-loop with $\phi = \frac{\rho+v}{\sqrt{2}} e^{i\theta/v}$

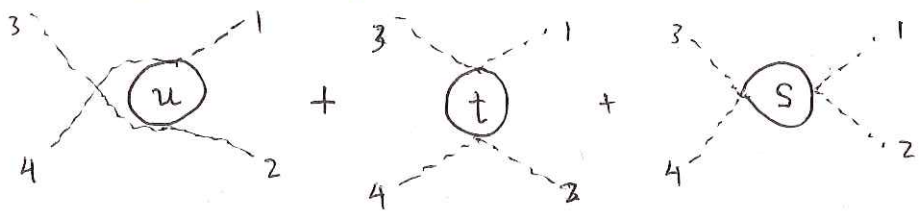
The key to computing the 1 loop amplitude correctly requires getting the Wick contractions and momentum interactions worked out correctly. Expanding about the vev, we find the interactions

$$\mathcal{L} \supset \frac{1}{2}(\partial_\mu\theta\partial^\mu\theta)\frac{\rho^2}{v^2} \Rightarrow$$

$$G_\theta = \text{- - - -} = \frac{i}{k^2 - m_\theta^2 + i\varepsilon} = \frac{i}{k^2 + i\varepsilon}$$

$$G_\rho = \text{———} = \frac{i}{k^2 - m_\rho^2 + i\varepsilon}$$

The oneloop contribution to the amplitude will come from 2nd order perterbation theory with this Lagrangian. There are 3 Feynman diagrams which contribute



In class, we worked out the s-channel contribution in some detail. I will repeat the set up to lead into the determination of the t- and u-channel contributions

$$i\delta A(s) = \frac{1}{2} \times 2 \times \left(\frac{-i}{2v^2}\right)^2 \times N_W \underbrace{(-ip_1)\cdot(-ip_2)\,(+ip_3)\cdot(+ip_4)} \cdot \mu^{4-d} \int \frac{d^dk}{(2\pi)^d} \frac{i}{(k+p_s)^2 - m^2 + i\varepsilon} \frac{i}{k^2 - m^2 + i\varepsilon}$$

2nd order perturbation theory

Time ordering of $\mathcal{L}_y \mathcal{L}_x$

Feynman Rule

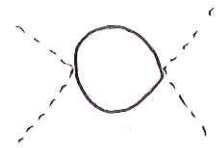
Wick contraction factor

Recal, the derivative interactions give

$$\partial_\mu \theta \to -ip_\mu \theta \quad \text{for incoming particles}$$

$$+ip_\mu \theta \quad \text{for outgoing particles}$$

s: $\langle \theta_3 \theta_4 | (\partial_\mu \theta \partial^\mu \theta \rho\rho)(\rho\rho \partial_\nu \theta \partial^\nu \theta) | \theta_1 \theta_2 \rangle$



2 choices

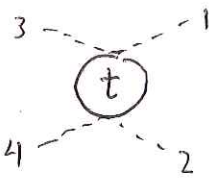
2 choices

1 choice

2 choices for this contraction

1 only 1 choice for schannel

$$\Rightarrow N_W = 2 \cdot 2 \cdot 2 = 8$$



2 choices (either 4 or 2)

1 choice

2 choices

: $\langle \theta_3 \theta_4 | (\partial_\mu \theta \partial^\mu \theta \rho\rho)(\partial_\nu \theta \partial^\nu \theta \rho\rho) | \theta_1 \theta_2 \rangle$

1 choice: in order to be t-channel, we must correlate 1-3, so once we chose 1, there was only 1 choice.

2 choices

$$\Rightarrow N_W = 8$$

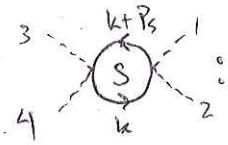
We will find the same counting for the u-channel also. We just need to make sure we get the momentum factors correct. Also, it is useful to have

$$P_1 \cdot P_2 = \frac{1}{2}(P_1+P_2)^2 - \frac{1}{2}P_1^2 - \frac{1}{2}P_2^2 = \frac{1}{2}s$$

$$P_1 \cdot P_3 = -\frac{1}{2}(P_3-P_1)^2 + \frac{1}{2}P_1^2 + \frac{1}{2}P_3^2 = -\frac{1}{2}t$$

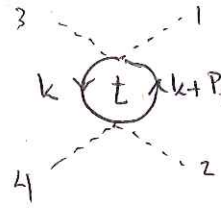
$$P_1 \cdot P_4 = -\frac{1}{2}u$$

a)



$$: \; i\delta A(s) = (i)^8(-)^4 \frac{8}{4v^4} \; P_1\cdot P_2 \; P_3\cdot P_4 \; \mu^{4-d}\int \frac{d^dk}{(2\pi)^d} \frac{1}{(k+P_s)^2-m^2+i\varepsilon} \frac{1}{k^2-m^2+i\varepsilon}$$

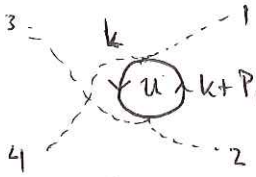
$$= \frac{1}{2}\frac{s^2}{v^4}\mu^{4-d}\int \frac{d^dk}{(2\pi)^d}\frac{1}{(k+P_s)^2-m^2+i\varepsilon}\frac{1}{k^2-m^2+i\varepsilon}$$



$$: \; i\delta A(t) = \frac{1}{2}\cdot 2\cdot\left(\frac{-i}{2v^2}\right)^2 N_W^t\,(-iP_1)(iP_3)\,(-iP_2)\cdot(iP_4)\,\mu^{4-d}\int\frac{d^dk}{(2\pi)^d}\frac{i}{(k+P_t)^2-m^2+i\varepsilon}\frac{i}{k^2-m^2+i\varepsilon}$$

$$= (i)^8(-1)^4\frac{8}{4v^4}\left(-\frac{1}{2}t\right)\left(-\frac{1}{2}t\right)\mu^{4-d}\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k+P_t)^2-m^2+i\varepsilon}\frac{1}{k^2-m^2+i\varepsilon}$$

$$= \frac{1}{2}\frac{t^2}{v^4}\mu^{4-d}\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k+P_t)^2-m^2+i\varepsilon}\frac{1}{k^2-m^2+i\varepsilon}$$



$$: \; i\delta A(u) = \frac{1}{2}\frac{u^2}{v^4}\mu^{4-d}\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k+P_u)^2-m^2+i\varepsilon}\frac{1}{k^2-m^2+i\varepsilon}$$

And we see the symmetry between the s, t, u channels we expect. We only have to solve 1 integral and just replace $P_s \to P_t \to P_u$

We solved this integral in class

$$\mu^{4-d}\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k+P)^2-m^2+i\varepsilon}\frac{1}{k^2-m^2+i\varepsilon}$$

$$= \mu^{4-d}\int_0^1 dx\int\frac{d^dk}{(2\pi)^d}\frac{1}{[k^2-\Delta(x,P)]^2}, \qquad \Delta(x,P) = x(x-1)P^2+m^2-i\varepsilon$$

$$= \int_0^1 dx\,\mu^{4-d}\,i\int\frac{d^dk_E}{(2\pi)^d}\frac{(-)^2}{[k_E^2+\Delta(x,P)]^2}$$

$$= i\int_0^1 dx\,\mu^{4-d}\frac{1}{(4\pi)^{d/2}}\frac{\Gamma(2-d/2)}{\Gamma(2)}\left(\frac{1}{\Delta(x,P)}\right)^{2-\frac{d}{2}}, \qquad \begin{aligned} d &= 4-2\varepsilon \\ d/2 &= 2-\varepsilon \\ 2-d/2 &= \varepsilon \end{aligned}$$

$$= i\int_0^1 dx \, \frac{\mu^{2\varepsilon}}{(4\pi)^2} (4\pi)^{\varepsilon} \frac{\Gamma(\varepsilon)}{1} \frac{1}{\Delta^{\varepsilon}(x,P)}$$

$$= \frac{i}{(4\pi)^2}\int_0^1 dx \left[\frac{1}{\varepsilon} - \gamma_E + \cancel{\ln}\, 4\pi + \ln\left(\frac{\mu^2}{x(x-1)P^2 + m^2}\right)\right]$$

$$\Rightarrow \delta A(P^2) = \frac{1}{2}\frac{(P^2)^2}{v^4}\frac{1}{(4\pi)^2}\left[\frac{1}{\varepsilon} - \gamma_E + \ln 4\pi - \left[\int_0^1 dx \, \ln\left(1 + x(x-1)\frac{P^2}{m^2}\right)\right] - \ln\frac{m^2}{\mu^2}\right]$$

$$= \frac{1}{2}\left(\frac{P^2}{4\pi v^2}\right)^2\left[\frac{1}{\varepsilon} - \gamma_E + \ln 4\pi - \ln\frac{P^2}{\mu^2} + 2 + 2\sqrt{\frac{4m^2}{P^2} - 1}\left(\tan^{-1}\sqrt{\frac{4m^2}{P^2}-1} - \frac{\pi}{2}\right) - \ln\frac{m^2}{P^2}\right]$$

$$= \frac{1}{2}\left(\frac{P^2}{4\pi v^2}\right)^2\left[\frac{1}{\varepsilon} - \gamma_E + \ln 4\pi - \ln\frac{m^2}{\mu^2} + 2 + 2\sqrt{\frac{4m^2}{P^2} - 1}\left(\tan^{-1}\left(\sqrt{\frac{4m^2}{P^2}-1}\right) - \frac{\pi}{2}\right)\right]$$

$$\boxed{\delta A = \delta A(s) + \delta A(t) + \delta A(u)}$$

b] To renormalize this interaction, we must have an operator of the form

$$\mathcal{L}_{ct.} = l \, \frac{\partial_\mu \theta \partial^\mu \theta \partial_\nu \theta \partial^\nu \theta}{(v^2)^2}$$, where l is an unknown coeficient.

Using the Wick contraction rules, it is straight forward to show

$$\langle \theta_3 \theta_4 | \, \partial_\mu \theta \partial^\mu \theta \, \partial_\nu \theta \partial^\nu \theta \, | \theta_1 \theta_2 \rangle = s^2 + t^2 + u^2$$

c) This operator contributes to the scattering amplitude

$$i\delta A_{c.t} = -\frac{i\ell}{v^4}\left(s^2+t^2+u^2\right)$$

$$\delta A_{ct} = -\ell\left(\frac{s^2+t^2+u^2}{v^4}\right)$$

The $\frac{1}{\varepsilon}$ poles will exactly cancel if we add this to the loop calculations. We can pick $\overline{MS}$ scheme to also cancel the $-\gamma+\ln 4\pi$

$$\delta A(s)+\delta A(t)+\delta A(u)+\delta A_{c.t.}\Big|_{1/\varepsilon \text{ contribution}} \qquad \frac{s^2+t^2+u^2}{v^4}\left[\frac{1}{2}\frac{1}{(4\pi)^2}\left(\frac{1}{\varepsilon}-\gamma+\ln 4\pi\right)-\ell\right]$$

$$\cancel{=\left[\frac{1}{2}\frac{1}{(4\pi)^2}-\ell\right]\frac{s^2+t^2+u^2}{v^4}\cdot\left[\frac{1}{\varepsilon}-\gamma+\ln 4\pi\right]}$$

$$\boxed{\Rightarrow\ \ell=\bar{\ell}+\frac{1}{2}\frac{1}{(4\pi)^2}\left[\frac{1}{\varepsilon}-\gamma+\ln 4\pi\right]}$$

d) $$\delta\bar{A} = \delta\bar{A}(s)+\delta\bar{A}(t)+\delta\bar{A}(u)+\delta\bar{A}_{c.t}$$

$$= \sum_{p^2=\{s,t,u\}}\left\{\left[2-\ln\frac{m^2}{\mu^2}+2i\sigma(p^2)\left(\tan^{-1}\left(i\sigma(p^2)\right)-\frac{\pi}{2}\right)\right]\frac{1}{2}\frac{1}{(4\pi)^2}-\bar{\ell}(\mu)\right\}\cdot\left(\frac{p^2}{v^2}\right)^2$$

$$\sigma(p^2)=\sqrt{1-\frac{4m^2}{p^2}}$$

$$\frac{\partial}{\partial\ln\mu^2}\delta\bar{A} = \sum_{p^2=\{s,t,u\}}\left(\frac{p^2}{v^2}\right)^2\left[\frac{1}{2}\frac{1}{(4\pi)^2}-\frac{\partial}{\partial\ln\mu^2}\bar{\ell}(\mu)\right]$$

$$\boxed{\Rightarrow\ \bar{\ell}(\mu)=\frac{1}{32\pi^2}\ln\frac{\mu^2}{\mu_0^2}}$$

4| $\rho(r) = N e^{-mr}$, $\int d^3r\, \rho(r) = 1$

a) What is N?

$$\int d^3r\, N e^{-mr} = 1$$

$$\int d^3r\, e^{-mr} = 4\pi \int_0^\infty dr\, r^2 e^{-mr}$$

$$= 4\pi \left(-\frac{\partial}{\partial m}\right)^2 \int_0^\infty dr\, e^{-mr}$$

$$= 4\pi \left(-\frac{\partial}{\partial m}\right)^2 \frac{1}{-m} e^{-mr}\Big|_0^\infty$$

$$= 4\pi \left(-\frac{\partial}{\partial m}\right)^2 \frac{1}{m}$$

$$= \frac{8\pi}{m^3}$$

$$\boxed{N = \frac{m^3}{8\pi}}$$

b) What is the form factor?

$$F(\vec{q}) = \int d^3r\, e^{i\vec{q}\cdot\vec{r}}\, \rho(\vec{r}) = ?$$

$$= 2\pi \int_0^\infty dr\, r^2 \underbrace{e^{iqr\cos\theta}}_{\int_{-1}^{1} d\cos\theta} N e^{-mr}$$

$$= 2\pi N \int_0^\infty dr\, r^2 e^{-mr} \frac{1}{iqr} e^{iqr\cos\theta}\Big|_{-1}^{1}$$

$$= 2\pi N \int_0^\infty dr\, r^2 e^{-mr} \frac{e^{iqr} - e^{-iqr}}{iqr}$$

$$F(\vec{q}) = 4\pi N \int_0^\infty dr \; r^2 e^{-mr} \frac{\sin(qr)}{qr}$$

$$= \frac{m^3}{2} \cdot \frac{2m}{(m^2+q^2)^2}$$

$$= \frac{1}{(1+q^2/m^2)^2}$$

c) $F(q) = 1 - \frac{1}{6} r^2 q^2 + \cdots$

$$G_E(-q^2) = (1 - q^2/m_0^2)^{-2}$$

$$\simeq 1 - 2\left(\frac{-q^2}{m_0^2}\right) + \cdots$$

$$\Rightarrow r^2 = \frac{12}{m_0^2}$$

$$r = \sqrt{\frac{12}{m_0^2}} = \sqrt{\frac{12}{.71}} \frac{1}{\text{GeV}}$$

$1973 \text{ MeV fm} = 1$

$$= \sqrt{\frac{12}{.71}} \frac{.1973 \text{ GeV fm}}{\text{GeV}}$$

$$\simeq 0.811 \text{ fm}$$