Thursday 13 March

- the correct "modern" way to do the

Gell-Mann — Oakes — Renner

is

$$m_q \rightarrow s(x)$$

then take functional derivatives of $Z^{\chi PT}$, Z^{QCD} so there is no $\int d^4x$ on both sides to ignore.

We were computing the scattering amplitude in χPT

$$A = -\frac{2}{f^2}\left[s - 2m_{\pi}^2\right]$$

At threshold, $s = 4m_{\pi}^2$

$$A = -\frac{2}{f^2} \cdot 2m_{\pi}^2 = -\frac{4m_{\pi}^2}{f^2}$$

In the chiral limit ($m_q \to 0$), the scattering Amplitude vanishes at threshold. This is again a consequence of the nature of Goldstone modes. (vanishing interactions at zero momentum).

We can also parameterize the scattering Amplitude in the partial wave expansion

$$A^I = 32\pi \sum_{\ell} (2\ell+1) P_{\ell}(\cos\theta)\, t_{\ell}^I(s)$$

$$t_\ell^I(s) = \frac{1}{2i\sigma(s)}\left[e^{2i\delta_\ell} - 1\right] \quad , \quad \sigma(s) = \sqrt{1 - \frac{4m^2}{s}}$$

Notice, our LO amplitude only has s-wave components. Non-trivial angular dependence is represented by

$t\ (s, t, u)$

$$t = (P_3 - P_1)^2 = (E_3 - E_1)^2 - (\vec{P}_3 - \vec{P}_1)^2$$

in C.O.M. $= -\vec{P}_3^2 - \vec{P}_1^2 + 2\vec{P}_3 \cdot \vec{P}_1$ C.O.M. = center of mass

$= -2|p|^2(1 - \cos\theta)$

whereas $s = (P_1 + P_2)^2 = (E_1 + E_2)^2 - (\vec{P}_1 + \vec{P}_2)^2$

$= (E_1 + E_2)^2$ $\quad$ $\vec{P}_1 + \vec{P}_2 = 0$ in C.O.M

in C.O.M.

$$A^I = 32\pi \sum_\ell (2\ell+1) P_\ell(\cos\theta)\, t_\ell^I(s)$$

$$= 32\pi\, t_0^I(s)$$

$$= 32\pi \frac{1}{2i\sigma(s)}\left[e^{2i\delta_0^I} - 1\right]$$

$$= 32\pi \frac{1}{\sigma_s} e^{i\delta}\left[\frac{e^{i\delta} - e^{-i\delta}}{2i}\right]$$

$$= 32\pi \frac{1}{\sigma_s} \frac{1}{e^{-i\delta}/\sin\delta}$$

$$A^I = \frac{32\pi}{\sigma(s)} \frac{1}{\cot\delta}$$

$$= \frac{32\pi}{\sigma(s)} \frac{1}{\frac{\cos\delta - i\sin\delta}{\sin\delta}}$$

$$= \frac{32\pi}{\sigma(s)} \frac{p}{p\cot\delta - ip}$$

$p = \sqrt{E^2 - m^2}$ center of mass momentum

$$= \frac{1}{2}\sqrt{4E^2 - 4m^2}$$

$$= \frac{1}{2}\sqrt{s - 4m^2}$$

$$= \frac{\sqrt{s}}{2}\sqrt{1 - \frac{4m^2}{s}}$$

$$= \frac{\sqrt{s}}{2}\sigma(s)$$

$$A^I = \frac{16\pi\sqrt{s}}{p\cot\delta^I - ip}$$

What is $p\cot\delta = ?$

Turns out, this is an analytic function which you can perform a Taylor expansion for near $p = 0$

Effective Range Expansion

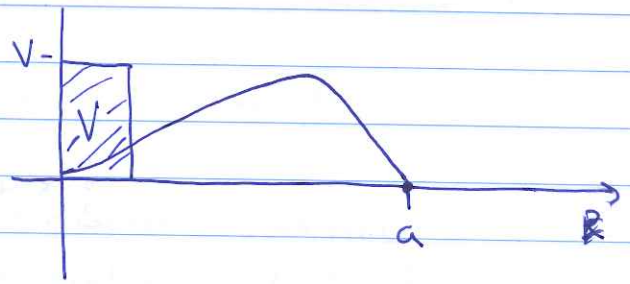
$$p \cot \delta = -\frac{1}{a} + \frac{1}{2} r p^2 + \cdots$$

a = scattering length

r = effective range

These coefficients parameterize the theory at low energy.

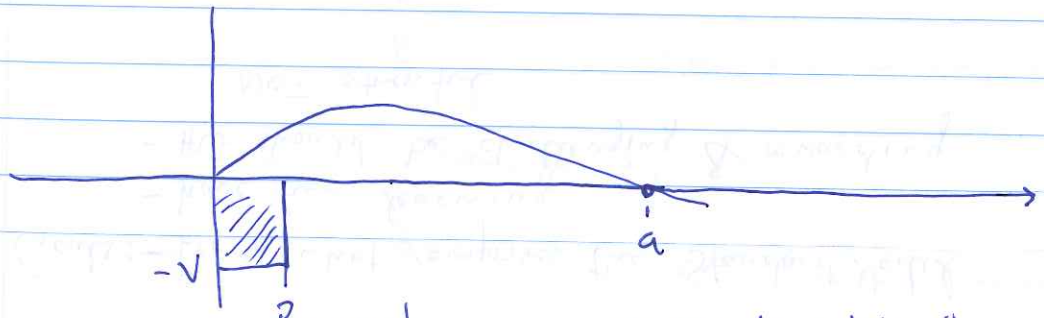
Suppose we have a repulsive interaction



a = location where zero energy wave-function crosses origin

$a > 0$

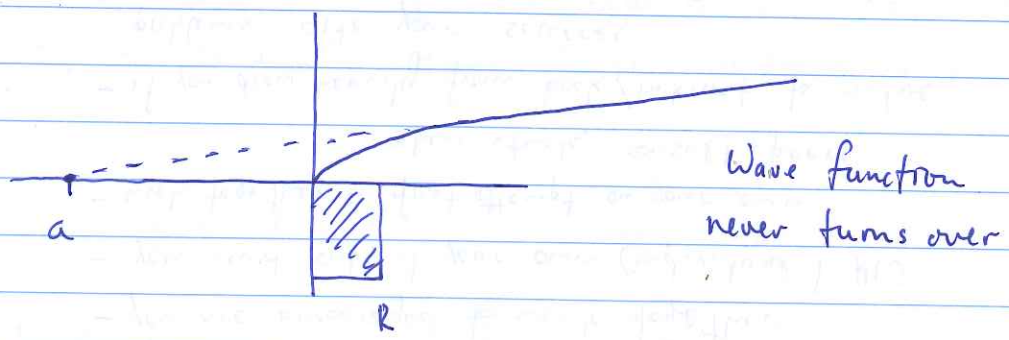
What about an attractive interaction?



Range of interaction

a can be arbitrarily different than R

You can also have an attractive interaction that does not support a bound state



$a < 0$

Notice, for a potential tuned to barely support a bound state $a \to \infty$
Then an ε weakening of V sends $a \to -\infty$

$a = \infty$ is called the unitary limit.

For zero energy scattering, in a theory described only with a scattering length ($r = 0$), the theory becomes conformal in the unitary limit ($a \to \infty$)
This is because there are no other length scales, and so if $a \to \infty$, there are no length scales.

The properties of such systems becomes universal because they don't depend on the details of the system.

$\frac{1}{a} < 0$: BCS Cooper pairs of long range correlation

$\frac{1}{a} = 0$: unitary limit, cross over

$\frac{1}{a} > 0$: BEC Tightly bound dimers

You may be familiar with this from AMO physics.

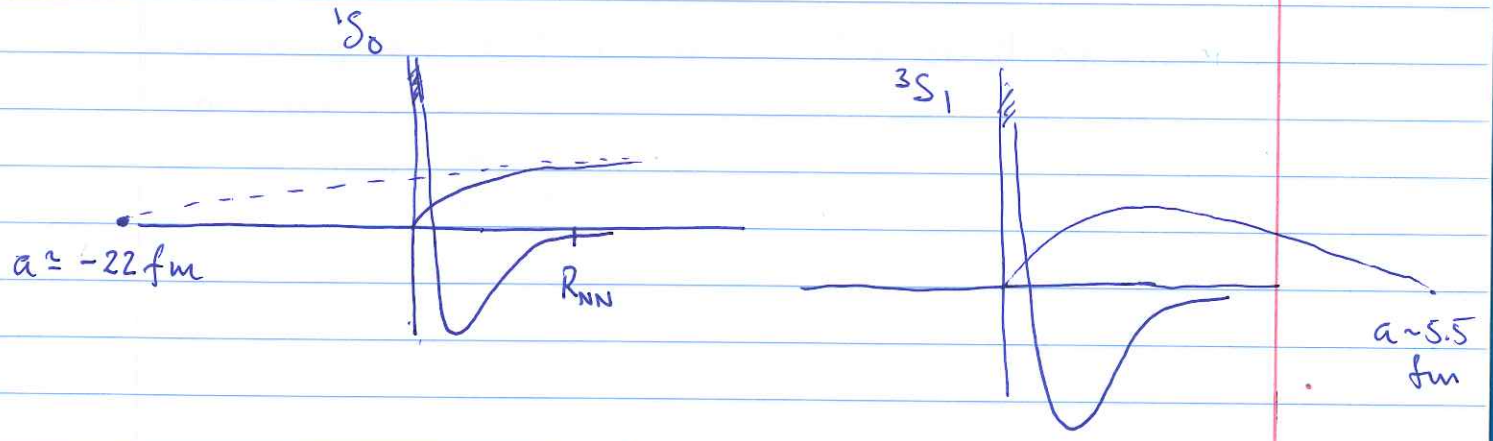
Revival of interest lately from experimental table-top ability to carefully tune these unitary systems.

This is also relevant in low-energy NN systems.

nn, np, pp
$^{2S+1}L_J = {}^1S_0$

$|\text{deuteron}\rangle = |np\rangle - |pn\rangle$
$^{2S+1}L_J = {}^3S_1$

In nuclear physics, the range of interaction is approximately $R_{NN} \sim \frac{1}{m_\pi} = \frac{1}{140\,\text{MeV}} = \frac{10}{7}\,\text{fm} \sim 1.4\,\text{fm}$



So unitary fermions are a semi-good model for low energy NN interactions

Back to our $\pi\pi$ example

$$A = \frac{16\pi\sqrt{s}}{p\cot\delta - ip} = \frac{-2}{f_\pi^2}\left[s - 2m_\pi^2\right]$$

at threshold $p\cot\delta \to -\frac{1}{a}$

$$s \to 4m_\pi^2$$

$$\frac{-4m_\pi^2}{f_\pi^2} = \frac{16\pi\sqrt{4m_\pi^2}}{-1/a}$$

$$m_\pi a = \frac{2}{8}\pi \cdot \frac{m_\pi^2}{8\pi^2 f_\pi^2} \qquad I = 2 \text{ channel}$$

In XPT, an EFT of QCD, the small expansion parameter is

$$\chi = \frac{m_\pi^2}{8\pi^2 f_\pi^2}$$

Nucleon Structure

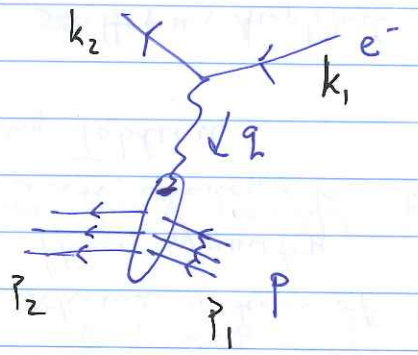
To resolve the structure of the proton, we often scatter electrons off them.

To the best of our ability (at least so far) the electron, as far as we can tell is point-like. It is therefore a very useful tool to probe the proton as the cross section already factorizes into a leptonic and hadronic part

$$d\sigma \sim L^{\mu\nu} W_{\mu\nu}$$

$L^{\mu\nu}$: Leptonic tensor, treated with perturbative QED

$W_{\mu\nu}$: Hadronic tensor



$$iA = ie\,\bar{u}(k_2)\gamma^\mu u(k_1)\,\frac{-ig_{\mu\nu}}{q^2}\,(-ie)\langle p_2 | J^\nu | p_1 \rangle$$

$\bar{u}(k_2)\gamma^\mu u(k_1)$: Dirac particle

$\langle p_2 | J^\nu | p_1 \rangle$: hadronic current matrix element

$$J^\nu = \bar{\psi}_{Prot}\gamma^\nu \psi_{Prot}$$

To normalize nucleon structure, we compare to scattering of point particles.

Rutherford Scattering: spinless particles

lab frame

$$\left.\frac{d\sigma}{d\Omega}\right|_R = \frac{\alpha_{f.s.}^2}{4E^2 \sin^4\frac{\theta}{2}} \frac{E'}{E}$$

Mott Scattering: point-like spin ½ scattering off spinless target

lab frame

$$\left.\frac{d\sigma}{d\Omega}\right|_M = \frac{\alpha_{f.s.}^2}{4E^2 \sin^4\frac{\theta}{2}} \frac{E'}{E} \cos^2(\theta/2)$$

Dirac Scattering: point-like spin ½ scattering off spin ½

lab frame

$$\left.\frac{d\sigma}{d\Omega}\right|_D = \frac{\alpha_{f.s.}^2}{4E^2 \sin^4\left(\frac{\theta}{2}\right)} \frac{E'}{E} \left[\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right]$$

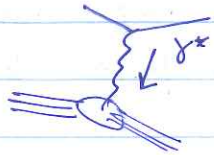
$$\frac{E'}{E} = \frac{1}{1 + \frac{2E}{M}\sin^2\frac{\theta}{2}}$$

arises from target recoil

$$q^2 = (E'-E)^2 - (\vec{k}_2 - \vec{k}_1)^2$$
$$= -4E'E\sin^2\frac{\theta}{2} = -Q^2$$

We are interested in modifications of these cross sections from nucleon structure. (QCD)

C.O.M energy of $\gamma^* N$ system



virtual photon ("massive") "off mass-shell"

$$(q+p)^2 = s = p^2 + 2p\cdot q + q^2$$

$$= M^2 + 2M\nu - Q^2 \qquad \nu = E' - E = \text{photon energy}$$

This form derived from lab frame, but s is Lorentz invariant

$W = \sqrt{s} \equiv$ "Mass" of ~~inner~~ state

$W < M + m_\pi \Rightarrow$ elastic scattering (nucleon stays nucleon) (Born)

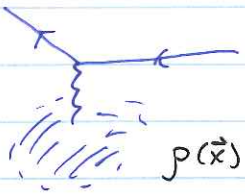
$W > M + m_\pi \Rightarrow$ inelastic scattering ($N \to \Delta (= N\pi)$, N^*, ⋮) (non-Born)

Lets first focus on elastic scattering.

$$\frac{d\sigma}{d\Omega} = \left.\frac{d\sigma}{d\Omega}\right|_M * |F(q)|^2$$

↑ form factor

For a static charge distribution,



$$\int d^3x\, \rho(\vec{x}) = \underline{1}$$

$$F(\vec{q}) = \int d^3x\, e^{i\vec{q}\cdot\vec{x}}\, \rho(\vec{x})$$

The form factor is just the Fourier Transform of charge density

If the charge density (proton) recoils, the the simple analogy between the charge density and form factor breaks down.

Considering only an electromagnetic current (no Parity violating weak current) the matrix element of the QED current in the proton is

$$\langle p_2 | J^\mu | p_1 \rangle = \bar{u}(p_2)\left[\gamma^\mu F_1(Q^2) + \frac{i\sigma^{\mu\nu} q_\nu}{2M} F_2(Q^2) \right] u(p_1)$$

$$F_1(0) = \hat{q} = \begin{cases} 1 & \text{proton} \\ 0 & \text{neutron} \end{cases}$$

$$F_2(0) = \kappa = \begin{cases} 1.79 & \text{proton} \\ -1.91 & \text{neutron} \end{cases}$$

Scattering cross-section

$$\frac{d\sigma}{d\Omega} = \left(\frac{E_e'}{E_e}\right)^2 \frac{|A|^2}{64\pi^2 M}$$

$$= \left.\frac{d\sigma}{d\Omega}\right|_M * \frac{E'}{E}\left[F_1(Q^2) + \tau\left\{ F_2(Q^2) + 2(F_1(Q^2) + F_2(Q^2))^2 \tan^2\frac{\theta}{2} \right\} \right]$$

$$\tau = \frac{Q^2}{4M^2}$$

It is convenient to define Sach's Electric & Magnetic form factors

$$G_E = F_1 - \tau F_2 \quad , \quad G_M = F_1 + F_2$$

$$\tau = \frac{Q^2}{4M^2}$$

$$F_1 = \frac{G_E + \tau G_M}{1+\tau} \qquad F_2 = \frac{G_M - G_E}{1+\tau}$$

~~$\frac{d\sigma/d\Omega}{d\sigma/d\Omega|_{NC}} = \frac{E'}{E}$~~

$$\frac{d\sigma}{d\Omega} = \frac{\alpha.f.s}{4E^2 \sin^4\frac{\theta}{2}} \frac{E'}{E} \left\{ \cos^2\frac{\theta}{2} \left(F_1^2 + \tau F_2^2 \right) + \sin^2\frac{\theta}{2} 2\tau \left(F_1 + F_2 \right)^2 \right\}$$

$$= \frac{\alpha.f.s}{4E^2 \sin^4\frac{\theta}{2}} \frac{E'}{E} \left\{ \cos^2\frac{\theta}{2} \frac{G_E^2 + \tau G_M^2}{1+\tau} + \sin^2\frac{\theta}{2} 2\tau G_M^2 \right\}$$

Rosenbluth cross section formula

$G_E(Q^2)$ = Electric form factor

$G_M(Q^2)$ = magnet form factor

Early Scattering experiments showed these two form factors are well approximated by a simple dipole model

$$G_E^p(Q^2) \sim \frac{G_M^p(Q^2)}{1+K^p} \sim \frac{1}{\left[1 + \frac{Q^2}{M_0^2}\right]^2}$$

$$M_0^2 \simeq 0.71 \text{ GeV}^2$$