Astro 201; Project Set #1

due friday, 2/03/2012

Problems focusing on the fundamental concepts of radiation (intensity, flux, opacity), and on solving on the radiation transfer equation under simple conditions. Emphasis is on the directional dependence of the specific intensity – no real concern is given yet to the detailed microphysics of the opacity and emission processes.

1) M87 and a "Continuum" of Stars

In the center of the Virgo cluster, about 16 megaparsecs away, lies the most massive elliptical galaxy in our neighborhood, M87 (see Figure 1). It consists of a pretty much spherical distribution of old stars spread out over a radius of around 10 kiloparsecs. At the center is a supermassive black hole – an active galactic nucleus (AGN) – which powers a relativistic jet.

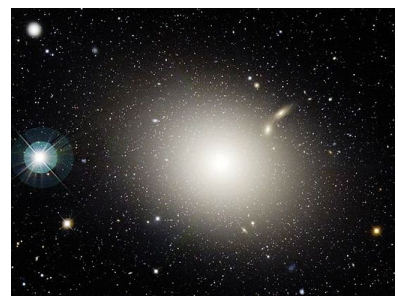

Figure 1: CFHT image of the massive elliptical galaxy M87.

M87 is a rich venue of radiative processes which we'll come back to later the class. For now, we'll just develop an overly simple model of the stellar radiation. Model the galaxy as $N = 10^{10}$ stars uniformly spread out in a sphere of radius $R_g = 10$ kiloparsecs. We'll approximate each star as a hard, completely opaque sphere with a mass, radius, and effective temperature equal to the solar values.

a) What is the average separation distance between the stars, in parsecs and in arcseconds? We'll assume that the smallest scale that can be resolved by an observer (or any other relevant galactic length scale for that matter) is significantly larger then this. In this case, we can imagine the stars to be a continuous medium (as we might do for, say, atoms in a gas).

b) Write down expressions for the cross-section, opacity, and extinction coefficient of our "stellar gas". How does the numerical value of this stellar opacity compare to a typical astrophysical value (e.g., the electron scattering opacity of ionized hydrogen)?

c) What is (numerically) the optical depth of the "stellar gas", $\tau_\star$, from the center of the galaxy to the edge? Given your result, you can assume from now on that the galaxy is optically thin[1], i.e., $\tau_\star \ll 1$.

[1] Of course, this is the optical depth to stars only; interstellar dust or gas spread throughout a galaxy may be optically thick at certain wavelengths. We won't worry about that now.

d) Write down an expression for the frequency integrated emission coefficient j (units ergs s^{-1} cm^{-3} str^{-1}) of the "stellar gas". What is the source function, and how does it compare to what you would expect for a thermal source?

e) Integrate the radiative transfer equation to determine the light profile of the galaxy – i.e., the specific intensity $I(p)$ as a function of the impact parameter p at which it is viewed at infinity (see figure 2). This quantity is closely related to the surface brightness profile of the galaxy.

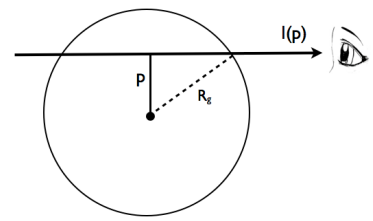

Figure 2: Diagram of the simple toy model of an ellipsoidal galaxy.

f) Integrate the specific intensity $I(p)$ over the entire projected area of the galaxy to determine the flux detected on earth. Compare to the result you expect by just summing up the luminosity of all the stars and dividing by $4\pi d^2$, where $d = 16$ Mpc is the distance to M87.

g) Let's calculate the radiation field throughout the galaxy produced by the stars. We'll solve the problem exactly in a second, but first orient yourself by writing down simple expressions (derived in a line or two) which give an order of magnitude estimate of the radiation energy density $u_\star$ and flux [2] $F_\star$ from stars in some typical part of the galaxy.

[2] Given the spherical symmetry here, when we say "the flux" we of course implicitly mean the radial component of the flux vector.

h) Now the exact solution: integrate the specific intensity to derive expressions for the mean intensity, $J_\star(r)$, the energy density $u_\star(r)$ and the flux $F_\star(r)$ of the radiation field from stars, all as a function of radius.[3]. To simplify the integration, you can limit your treatment to the inner regions of the galaxy where $r \ll R_g$. If you feel like tackling the full integral, you might consult the online mathematica integrator or some other tool.

[3] You'll probably want to use the "law of cosines".

i) Plot your findings for $u_\star(r)$ and $F_\star(r)$ as a function of radius. Derive the values at the galactic center ($r = 0$) and explain why your result is obviously correct at this point. Also explain why it makes sense that $u_\star(r)$ decreases with radius, while $F_\star(r)$ *increases* with radius.

j) The radiation field from stars will heat interstellar gas and dust. Define a radiation "temperature" T by setting your calculated radiation energy density $u_\star(R)$ equal to the energy density of a blackbody, aT^4. If interstellar dust in the galaxy was warmed to approximately this temperature[4], at around what wavelengths would you look for the dust emission?

[4] This is the temperature gas or dust would be heated to if it was in equilibrium with the stellar radiation and the opacity was wavelength independent

Project #2: Stellar Limb Darkening and Exoplanet Transits

THE FIRST DETECTION OF AN EXTRA-SOLAR PLANET TRANSITING ITS STAR was HD 209458b (Charbonneau et al. 2000) – a hot, jupiter-sized planet orbiting $\sim$ 0.05 AU from its host star. The orientation of the system is such that the planet briefly eclipses the star when it passes in front of it, as seen in the original photometric data:

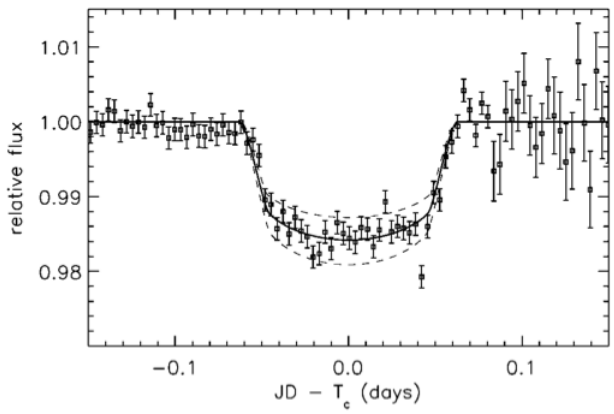

Figure 3: First data on a transiting expolanet, HD209568b, from Charbonneau et al., 2000 . Filled circles show the data binned into 5 minute intervals. The lines show model light curves (which include the limb-darkening effect) a summing different planetary radii.

You'll notice something interesting in the data – the dip in the transit light curve is curved, not flat-bottomed. This is because of the limb-darkening effect – i.e., light emitted from the edge of a star is less intense than light coming from the center. This effect has been known for over a hundred years from observations of the sun (see Figure 4). To accurately determine the properties of a transiting exoplanet (e.g., orbital inclination, planet radius) stellar limb-darkening must be taken into account.

Modeling stellar limb-darkening is one of the classic problems in theoretical astrophysics, going back to Schwarzchild's solution in 1906 and subsequent work by Eddington and Milne.[5] In this project, we'll derive a workable model for exoplanet transit light curves by solving the radiation transport equation (ala Schwarzchild) to account for limb darkening.

[5] You can find Milne's 1921 paper on limb darkening here .

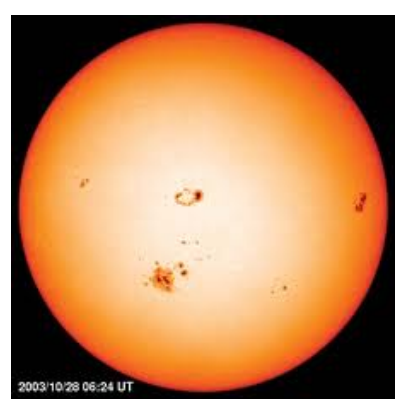

Figure 4: The sun, showing limb darkening (also limb-reddening)

The Limb Darkening Problem

Limb darkening arises because the light from the edge of the star is seen at a slant with respect to the normal vector (see figure 5). To quantify the effect, we will need to solve the radiation transfer equation to determine the dependence of the specific intensity on the angle ($\mu = \cos\theta$) at which it is observed. The atmospheres of stars like the sun are very thin compared to the radius, so we can use the

radiative transfer equation in plane parallel coordinates:

$$\mu \frac{\partial I_\nu(\tau_z, \mu)}{\partial \tau_z} = I_\nu(\tau_z, \mu) - S_\nu(\tau_z, \mu) \tag{1}$$

We might think of this as not just one equation, but an infinite number of differential equations – one for each different frequency ν and direction μ. To make the problem tractable, we'll have to make several typical approximations.

Approximation #1: The opacity is wavelength independent (grey)

In this case, we can drop the ν subscripts in equation 1 and only use quantities that have been integrated over all frequencies. Of course, we can not also drop the μ dependence (i.e., assume the radiation is isotropic) as that would miss the essential physical effect behind limb-darkening. We can simplify life, though, by adopting a relatively simple form for the angular dependence:

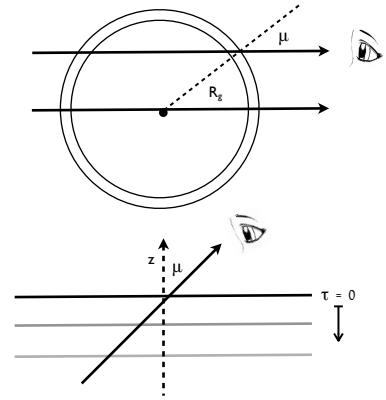

Figure 5: Geometry for the problem. To find the specific intensity observed near the edge of a spherical star (top) we solve the problem for a plane parallel atmosphere (bottom) and find the intensity emerging at the equivalent angle to the normal, $\mu = \cos\theta$.

Approximation #2: The angular distribution of I can be described by

$$I(\tau_z, \mu) = I_0(\tau_z) + I_1(\tau_z)\mu \tag{2}$$

where we require $I_0 > I_1 > 0$ for all τ_z (so that the intensity never goes negative). This expression is obviously an over-simplification of the angular dependence of I, which could in principle be a complicated function[6] of μ. By specifying this simple form, however, we will no longer need to solve an infinite number of transport equations; instead, we'll only need two equations to determine the two unknowns: I_0 (the isotropic part) and I_1 (the anisotropic part).

[6] At least equation 2 captures one crucial feature: more radiation is moving out of the star ($\mu > 0$) then inward ($\mu < 0$). Our expression actually represents the appropriate limit in the optically thick diffusion regime, as we will discuss later.

a) Write down expressions (in terms of I_0, I_1) for the mean intensity $J(\tau_z)$ and the astrophysical flux $F(\tau_z)$. For fun, also write down the radiation energy density $u(\tau_z)$ and pressure $P(\tau_z)$. What is the ratio P/u and how does it compare to that of isotropic radiation?[7].

[7] The ratio $f = P/u$ is called the Eddington factor and the assumption $f = 1/3$ is called the Eddington approximation. It holds not only for isotropic radiation, but also for any $I(\tau_z, \mu)$ that only depends on odd powers of μ, such as ours.

Now we'll solve the radiation transfer equation. Since there are two unknowns (I_0 and I_1) we'll use two moments of the transfer equation to form a complete system of equations:

b) Take the zeroth moment of the radiation transport equation (i.e., integrate equation 1 over all angles) and show that the source function is given by $S = J$ provided we make another assumption:

Approximation #3: The atmosphere is in radiative equilibrium[8]*, such that the flux,* $F_0 = \sigma T_{\rm eff}^4$, is constant with τ_z.

[8] In the early days, it was not known whether the atmosphere of the sun was in radiative equilibrium or whether it was convective. By solving this problem we'll show (as Schwarzchild did in 1906) that the observed limb darkening implies that it is indeed radiative.

c) Take the first moment of the transport equation and use it to find an expression for $I(\tau, \mu)$ which depends on two constants: $T_{\rm eff}$, and some constant of integration.

To determine the constant of integration, we need to specify some kind of boundary condition. We'll make another approximation:

Approximation #4: At the surface of the star ($\tau = 0$) the inward directed flux (i.e. integrated only over the directions $-1 < \mu < 0$) is zero.

This is one of several approximate boundary conditions that might be chosen. It captures our intuition that the light is streaming out of the stellar surface, but none is streaming inward.

d) Determine the integration constant to complete the solution. Now that you know everything, you should be able to plot the limb darkening law for the relative emergent intensity as a function of μ. Compare[9] your solution to the early observations taken by Muller (1897) shown in Figure 6. Does your calculation indeed suggest that the solar atmosphere is radiative?

[9] You might also compare with your solution for problem 1d, which gives the "limb-darkening" of an *optically thin*, homogenous radiating sphere.

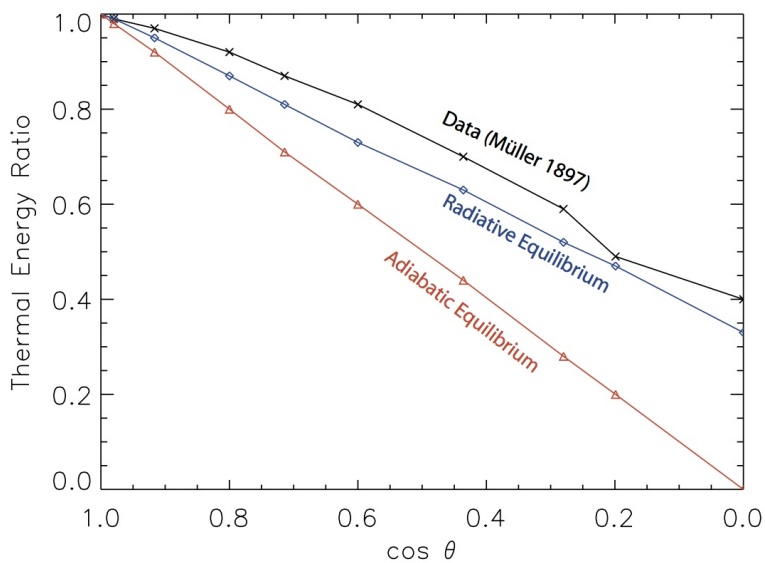

Figure 6: Solutions for the limb darkening law assuming a radiative and a convective atmosphere. Also shown are early observations from Muller (1897). Adapted from K. Schwarzschild (1906) by J. Aufdenberg . The y-axis plots the ratio $I(\tau = 0, \mu)/I(\tau = 0, \mu = 1)$. Note that Schwarzschild made slightly different approximations than we have, and so derived a slightly different solution ($I \propto \mu + 1/2$, shown as the blue line). Your solution will actually fit the data better.

Explaining the Planetary Transit Data

At last we can model (approximately) the light curve of HD209568b. Assume the orbit has an inclination of 90° (i.e., is exactly edge on as viewed from earth) and that the duration of the transit is 0.12 days. For our purposes, you can make the approximation that the radius of the planet R_p is much less than that of the star, $R_\star$ (i.e., no need to integrate over the size of the planet). In this case, you won't get the sharp edges of the light curve quite right, when the planet moves on or off the stellar disk (the ingress and egress marked by w on figure 7).

e) Write an expression for the transit light curve – i.e., the observed flux as a function of time (no need to worry about the overall normal-

ization of the flux; just the relative value as in Figure 3 is fine). Plot your light curve and compare by eye to the observed data in Figure 3.

Comment: You will notice that your light curve has significantly more curvature than the data shown in Figure 3. That is because the inclination of HD209568b is not exactly 90° and the planet does not pass exactly along the mid-plane, rather it grazes its host star, as illustrated in figure 7. **Bonus:** By thinking a little more about the geometry, work out how to use a measurement of the curvature parameter c (defined in figure 7) to determine the inclination of the orbit, based on your equation for limb-darkening (and assuming the stellar and orbital radius are given by other means).

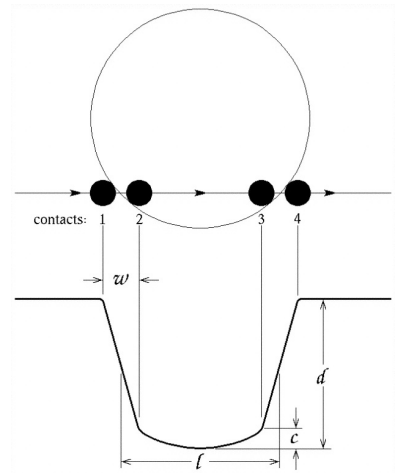

Figure 7: Schematic of the shape of the light curve from an expolanet transit, taken from Brown et al., (2001) .

Explaining the Sun's Red Edge

Notice in Figure 2 that the edge of the sun also looks redder then the center, which is a closely related effect. Let's calculate how much redder. To do so we'll make a further assumption:

Approximation #5: The atmosphere is in LTE, so that the source function $S(\tau)$ is everywhere equal to the blackbody value.

e) Write an expression for the temperature of the atmosphere as a function of τ_z (and depending on $T_{\rm eff}$)[10]. Now assume that when you look at the star at some angle μ, you see roughly a blackbody spectrum at the temperature at a *line of sight* optical depth of $\simeq 2/3$. If the effective temperature of the Sun is $T_{\rm eff} = 5800$ K, how much "cooler" is the spectrum of radiation from the edge ($\mu = 0$) compared to the center ($\mu = 1$)?

[10] Your (approximate) solution for the temperature structure of a radiative atmosphere can be quite a useful one for certain problems.

Comment: If we wanted to derive a more accurate solution for the limb darkening problem, we could write the angular dependence of $I(\tau_z, \mu)$ as an expansion in Legendre polynomials:

$$I(\tau_z, \mu) = \sum_{i=0} I_i(\tau_z) P_i(\mu) \tag{3}$$

What you have done is to solve the problem when keeping only the first two terms in this series. For each additional higher order term you include, a new unknown is introduced and you would need to take another higher moment of the transfer equation to complete the system of equations.[11]

[11] See Chandrasekhar's 1960 book for more. Since the radiation field does not usually oscillate very rapidly with μ, you could probably get away with only the first few terms in the series.

Comment: There are other approximate ways of representing the angular dependence of I. In the *two-stream approximation* we assume that the intensity has one value going up, and another value going down:

$$I(\tau_z, \mu) = I^+(\tau_z)\Theta(0, \pi/2) + I^-(\tau_z)\Theta(\pi/2, \pi) \tag{4}$$

where $\Theta(x_1, x_2) = 1$ for $x_1 < x < x_2$ and 0 otherwise. **Bonus:** If you felt like it, you should be able to use this functional form to solve the transport equation using a similar approach described above.

The generalization of the two stream approximation would be an n-stream approximation:

$$I(\tau_z, \mu) = \sum_i^n I_i \Theta(\mu_i, \mu_i + \Delta\mu_i) \tag{5}$$

Where we would choose the angles μ_i and $\Delta\mu_i$ to give a favorable spacing of the n rays. You should easily convince yourself that $J = \sum_i I_i \Delta\mu_i / 2$. Assuming radiative equilibrium ($J = S$) the set of radiative transfer equations then become:

$$\mu_i \frac{\partial I_i(\tau_z)}{\partial \tau_z} = I_i - \frac{1}{2} \sum_i^n I_i \Delta\mu_i \tag{6}$$

This is just a system of n coupled linear differential equations which in principle can be solved (numerically) to determine the $I_i(\tau_z)$. The higher n, the more accurate the solution. This sort of method of *discrete ordinates* is a common approach to solving radiation transport problems on computers.

Bonus (optional) problem – Astro 201: **the real world**

Increase your car remote range with your face!

We've all had the experience of being lost in a parking lot, trying to find our car by clicking the key remote repeatedly and hoping to hear a beep or see a flash. Well, now an old friend from high school presents the following unlikely lifehack[12]: **Hold your car remote against your chin when you push the button and you will increase its range.**

That sounded ridiculous, so I tried it myself. I found the maximum distance at which my remote would work (about 30 feet) then I held the remote to my chin and took several steps back. It worked! I was able to get several feet more range. Try it yourself!

So I had to hand it to my friend. Still, his explanation of the effect ("the water in your head acts like a huge radio antennae", boosting the signal) seemed pretty bogus to me. What do you, Astro201 student, think the *real* explanation for the extended range is? What is the theoretical maximum factor by which you might boost your range using this kind of trick?

[12] You can read his blog post ; you can also check out some of his clever ideas on how to defeat the government